

Smooth Shape-Aware Functions with Controlled Extrema

Alec Jacobson¹

Tino Weinkauf²

Olga Sorkine¹

¹ETH Zurich

²MPI Saarbrücken



INTERACTIVE GEOMETRY LAB

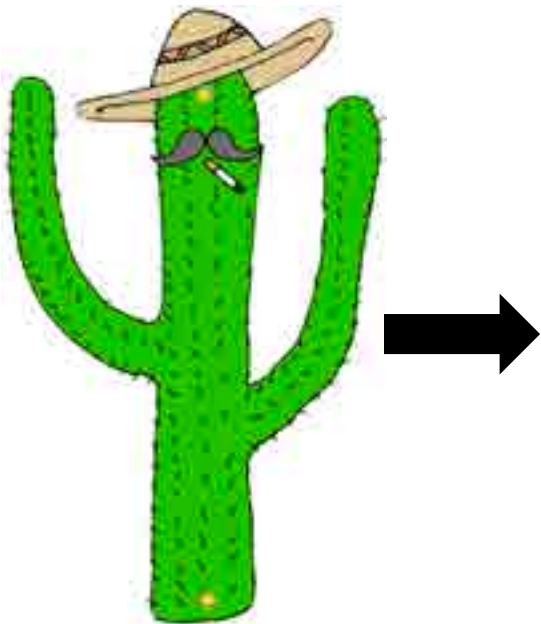
August 9, 2012



Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

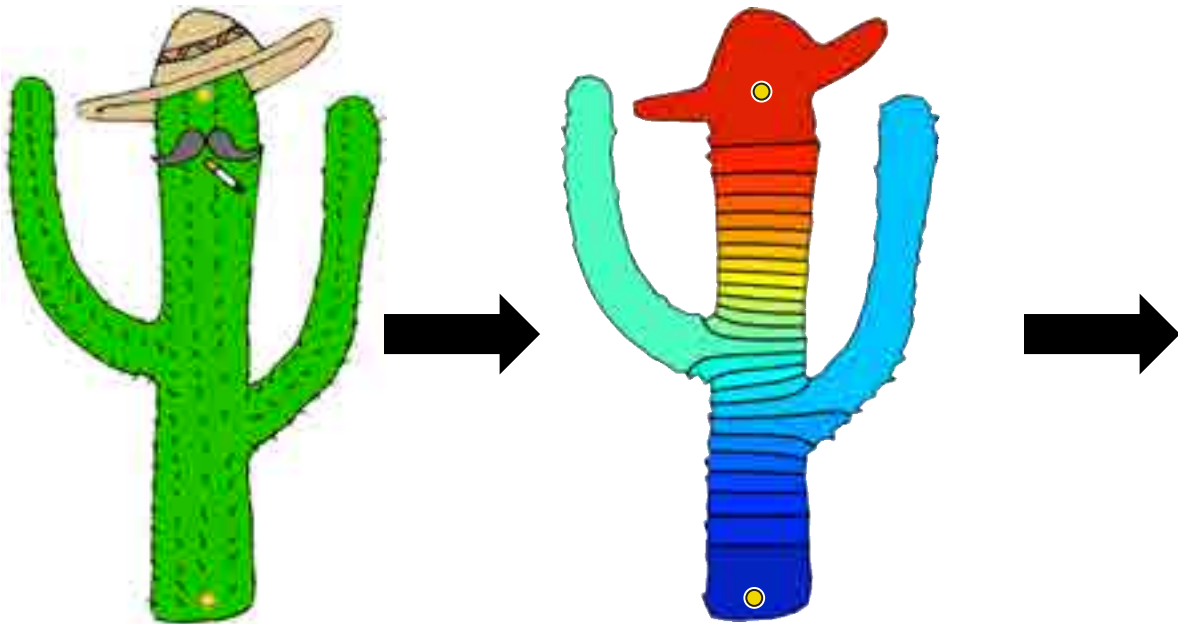
Real-time deformation relies on smooth, shape-aware functions

input shape + handles



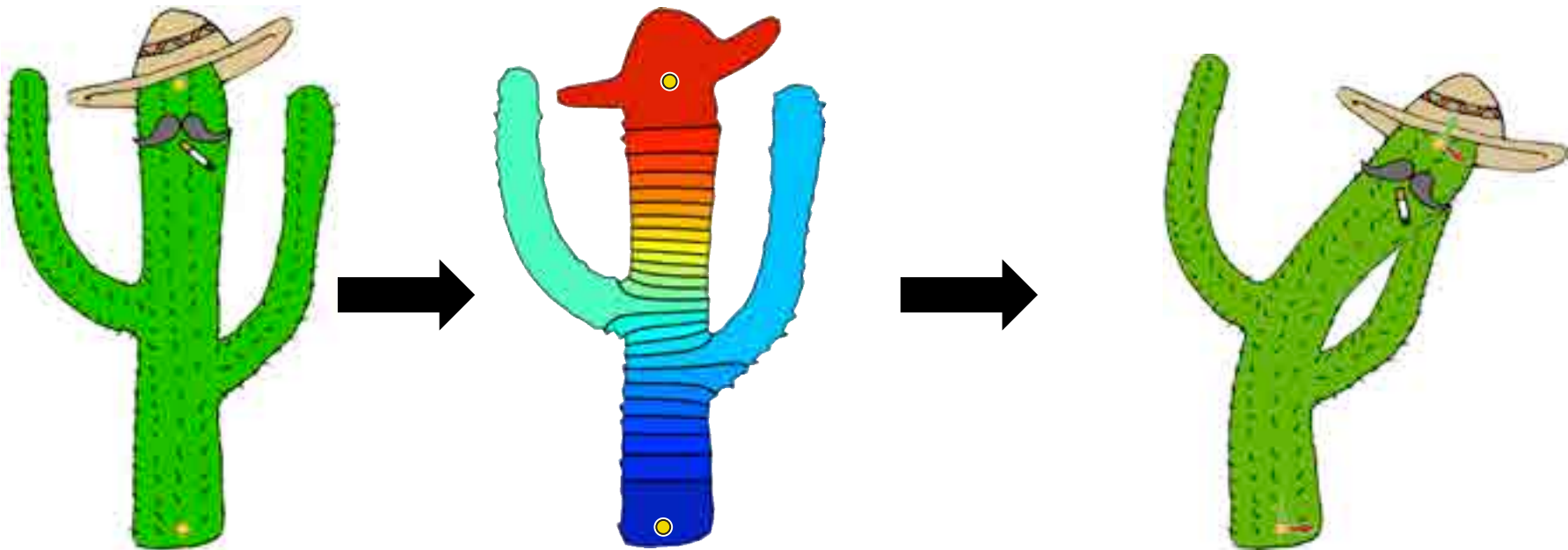
Real-time deformation relies on smooth, shape-aware functions

precompute weight functions

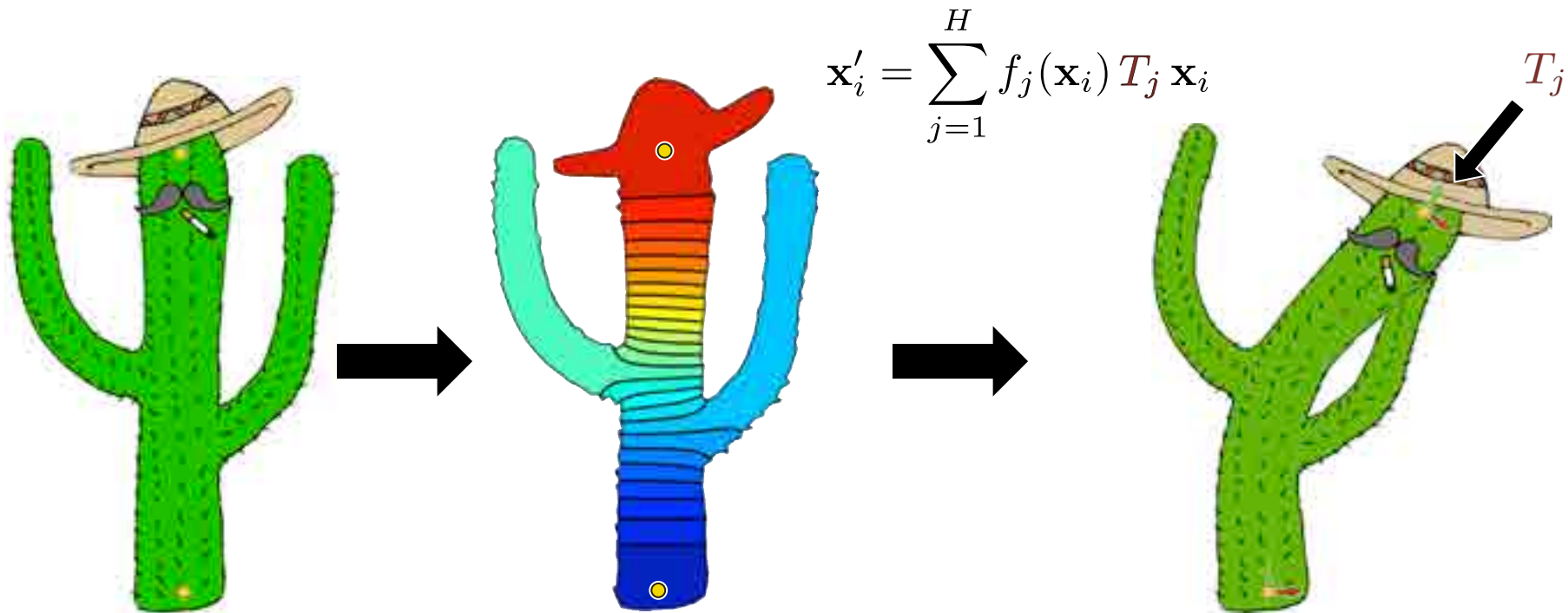


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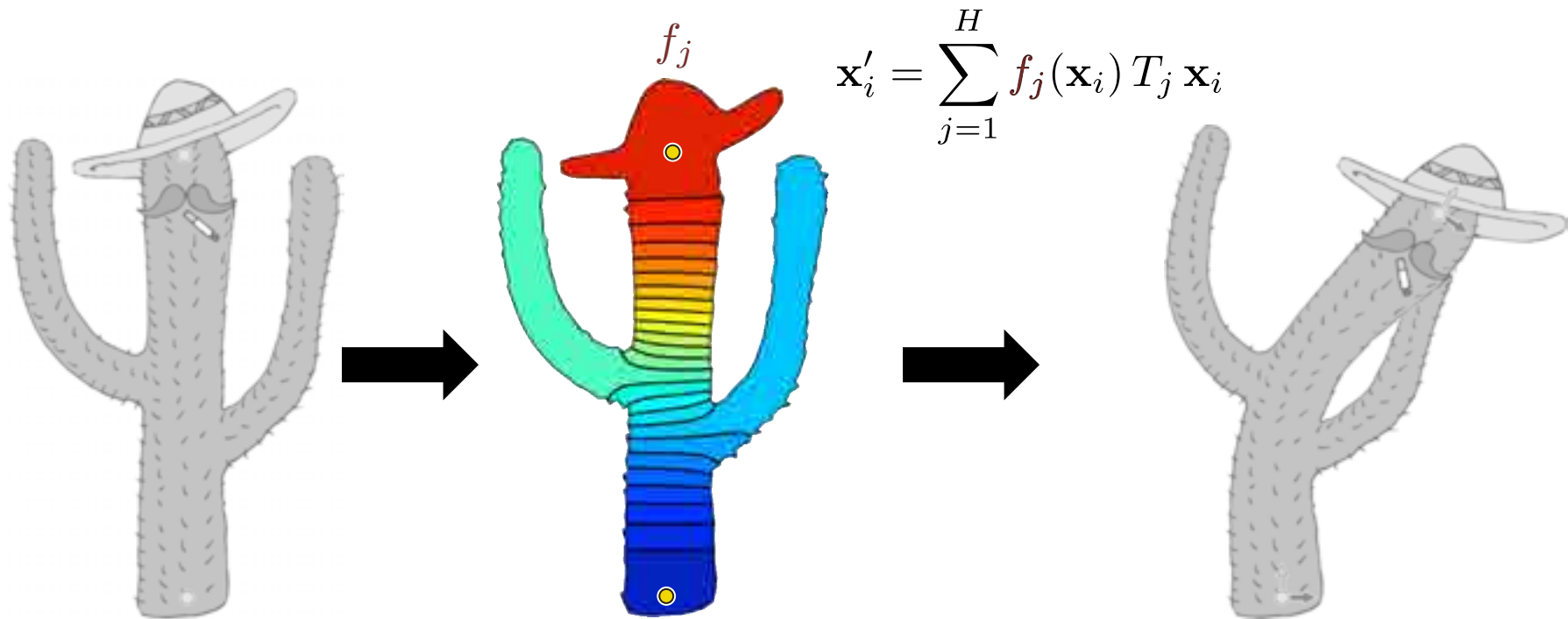
deform handles $\rightarrow$ deform shape



Real-time deformation relies on smooth, shape-aware functions

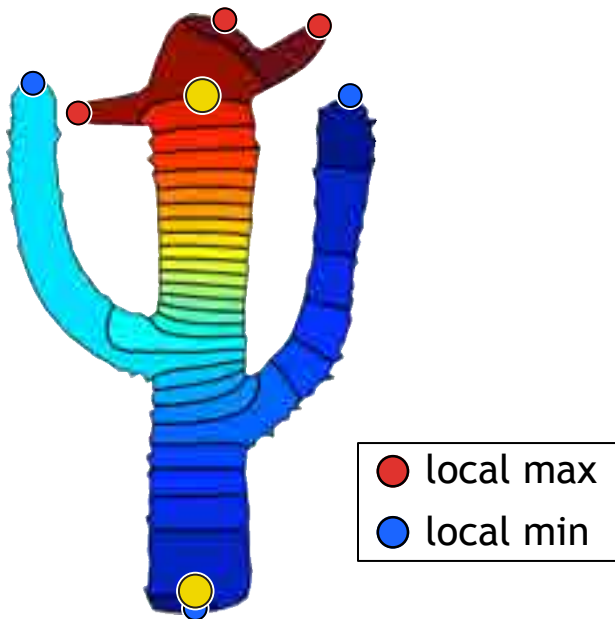


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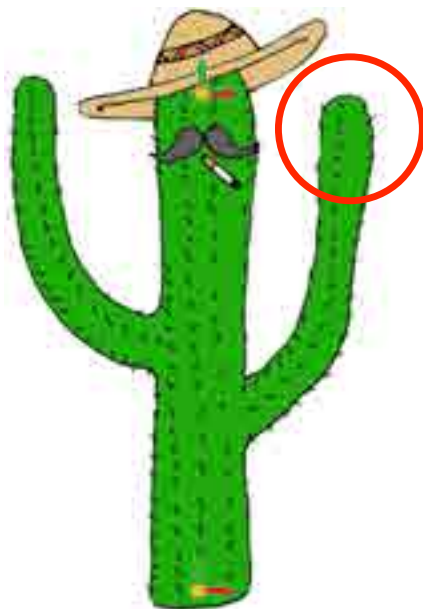


Spurious extrema cause distracting artifacts

unconstrained Δ^2
[Botsch & Kobbelt 2004]



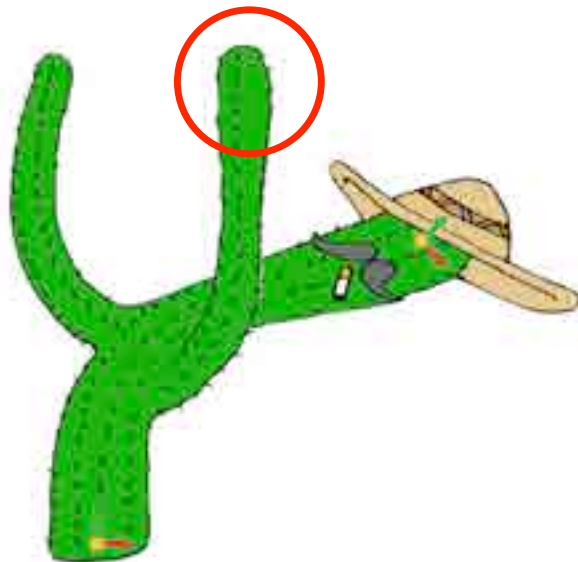
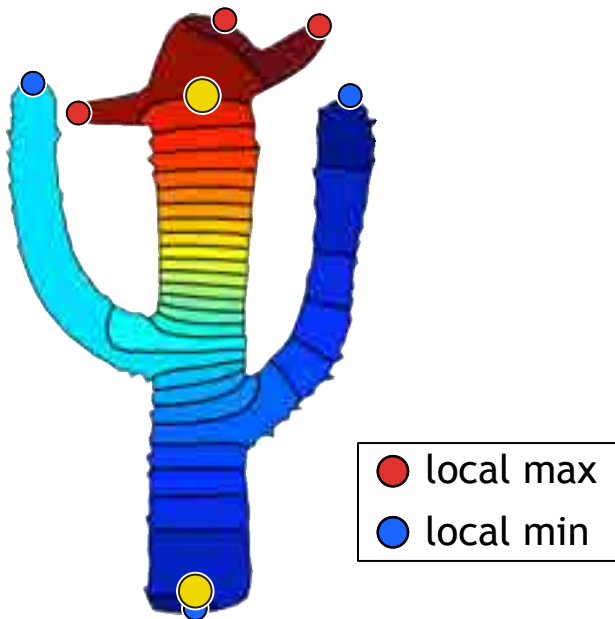
$$\mathbf{x}'_i = \sum_{j=1}^H f_j(\mathbf{x}_i) T_j \mathbf{x}_i$$



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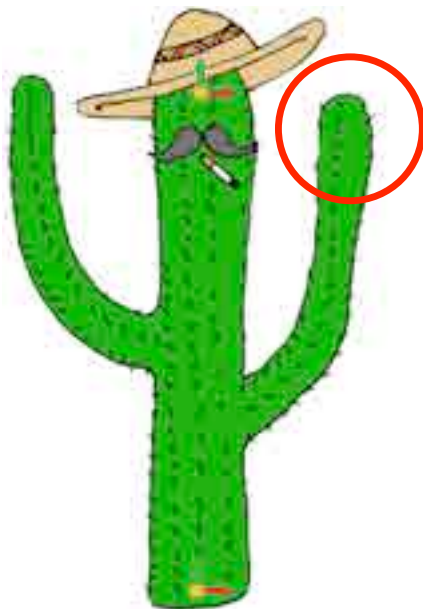
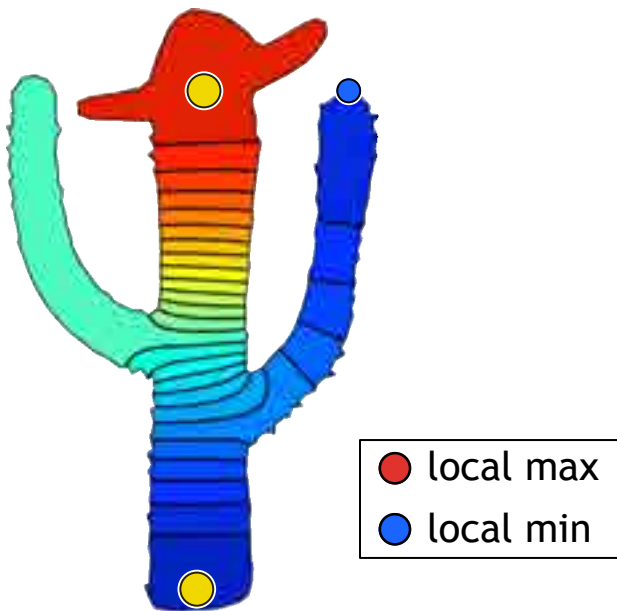
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Bounds help, but don't solve problem

bounded Δ^2
[Jacobson et al. 2011]

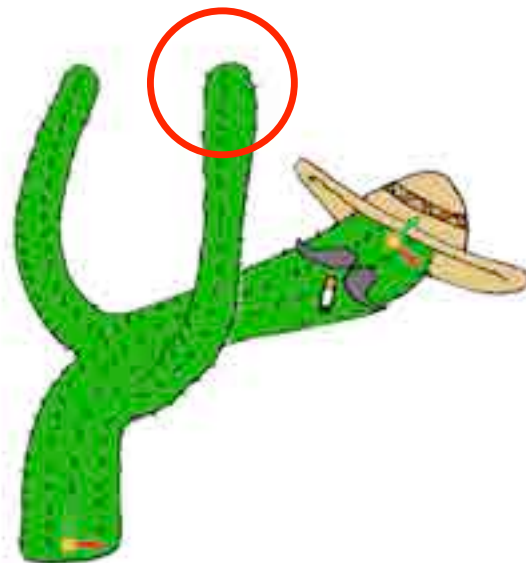
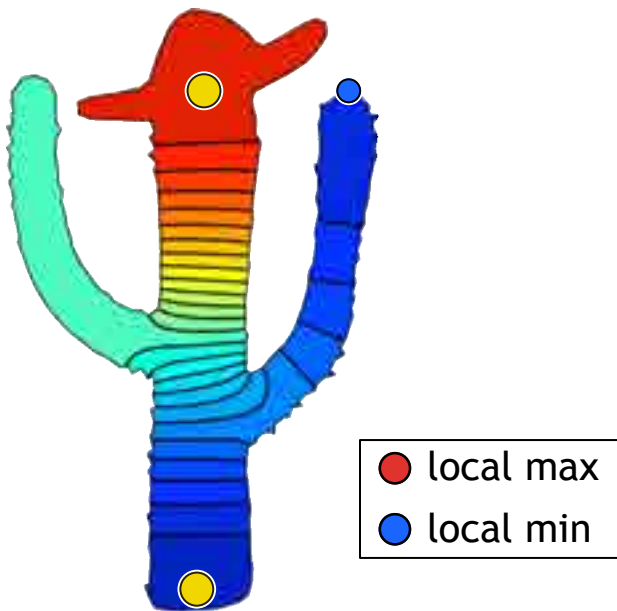
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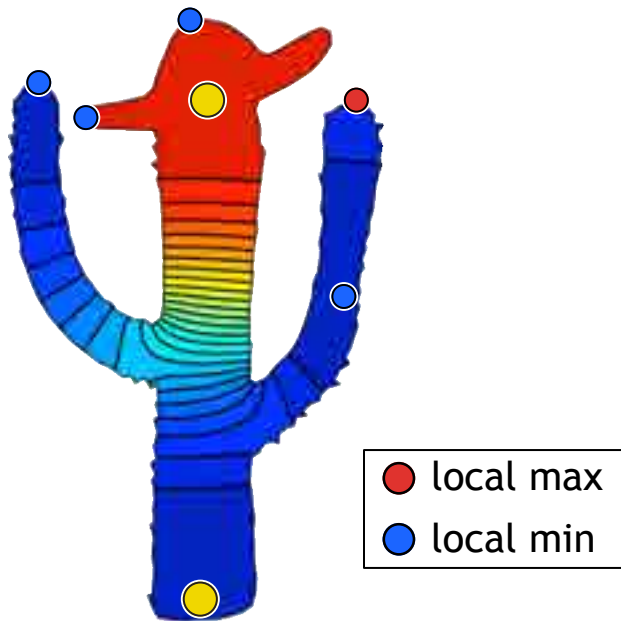
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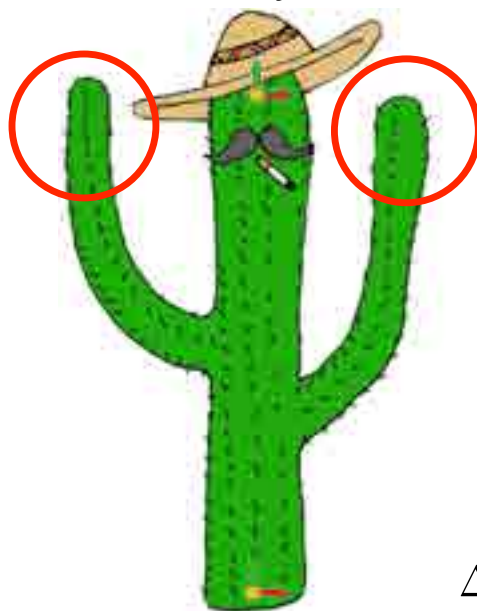


Gets worse with higher-order smoothness

bounded Δ^4
[Jacobson et al. 2011]



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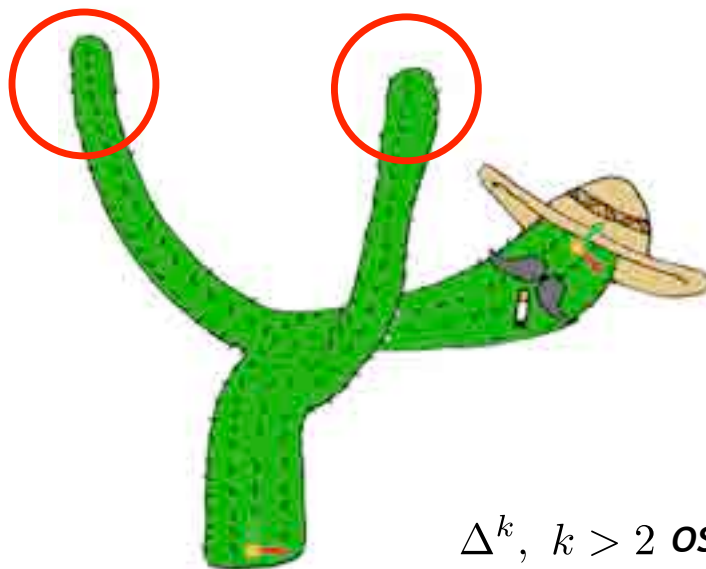
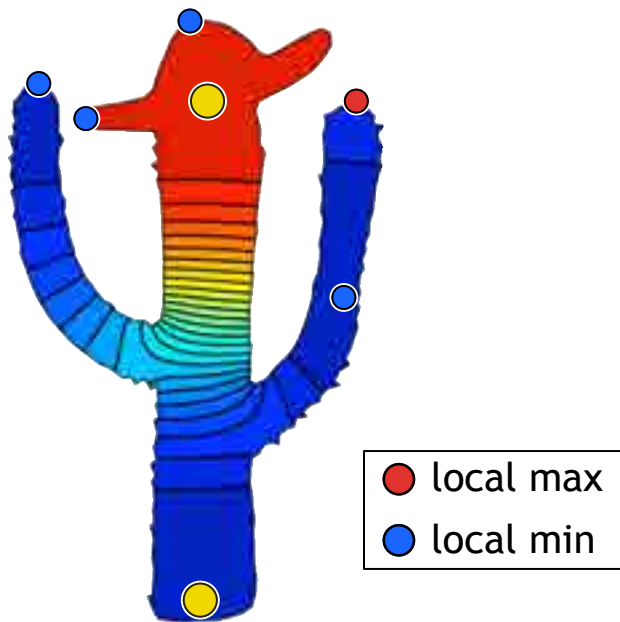


$\Delta^k, k > 2$ oscillate too much

Gets worse with higher-order smoothness

bounded Δ^4
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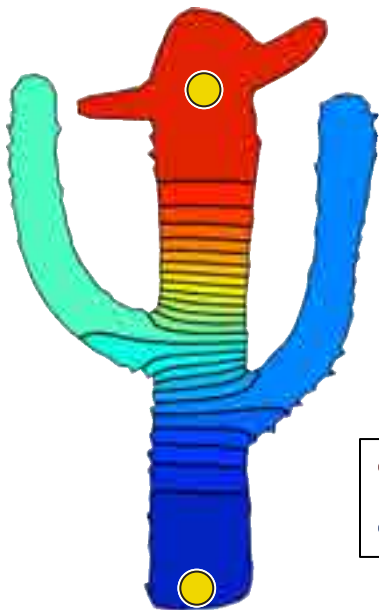
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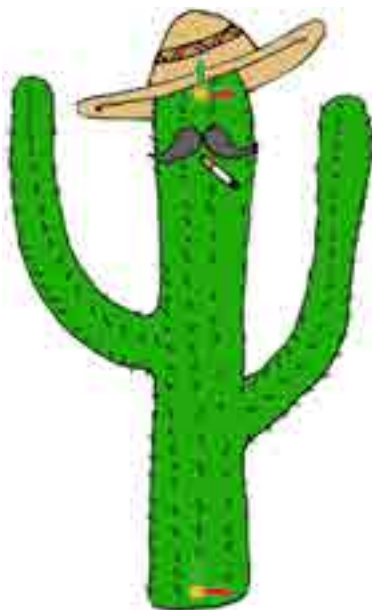
We *explicitly* prohibit spurious extrema

our Δ^4



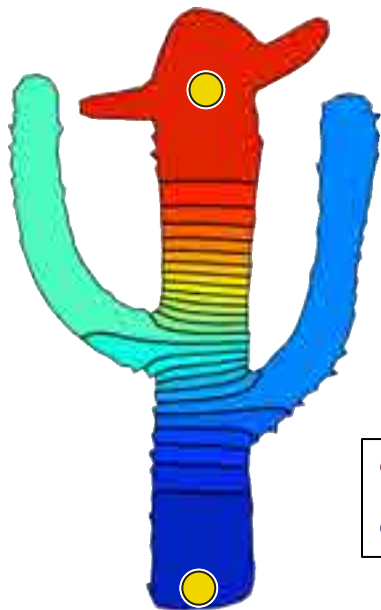
● local max
● local min

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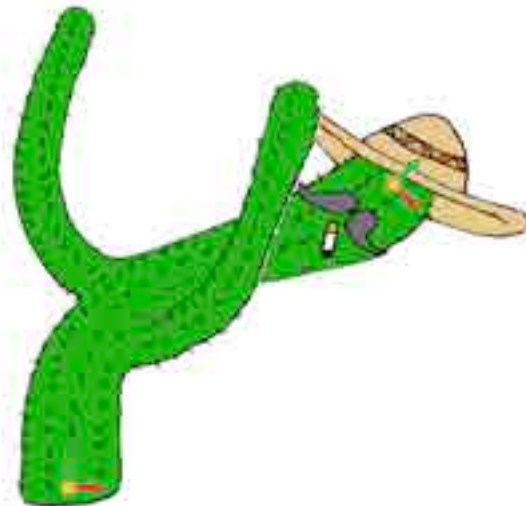


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$$\mathbf{x}'_i = \sum_{j=1}^H f_j(\mathbf{x}_i) T_j \mathbf{x}_i$$



Same functions used for color interpolation

$$\mathbf{x}'_i = \sum_{j=1}^H f_j(\mathbf{x}_i) T_j \mathbf{x}_i$$

Same functions used for color interpolation

$$\mathbf{c}_i = \sum_{j=1}^H f_j(\mathbf{x}_i) \mathbf{c}_j$$

Same functions used for color interpolation

unconstrained Δ^2
[Finch et al. 2011]

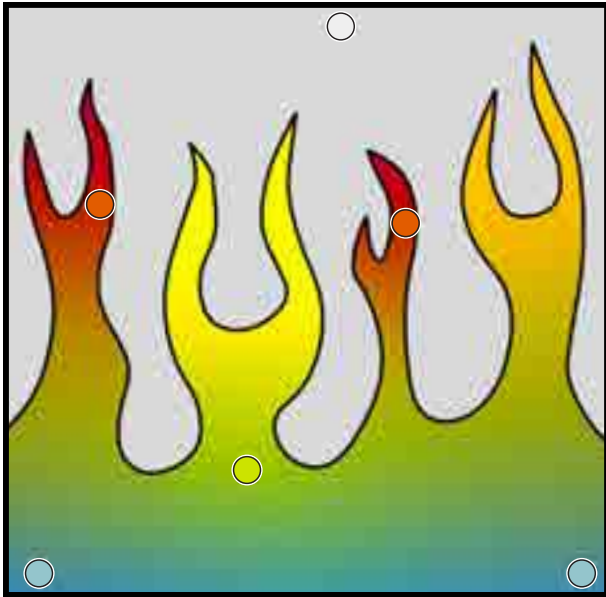
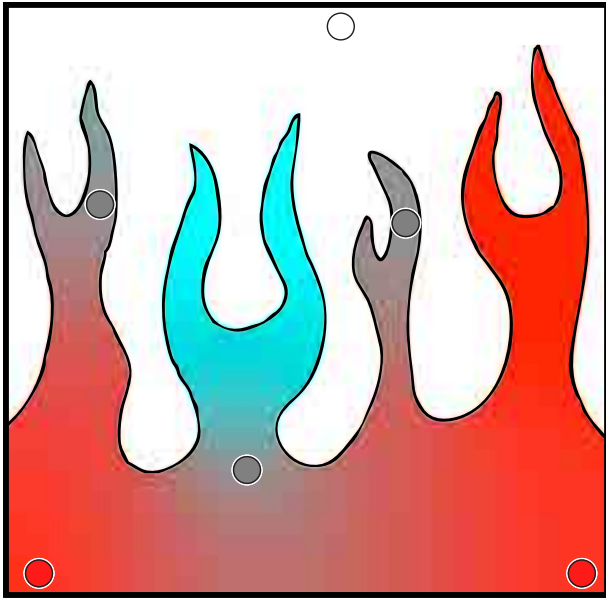


Image courtesy Mark Finch

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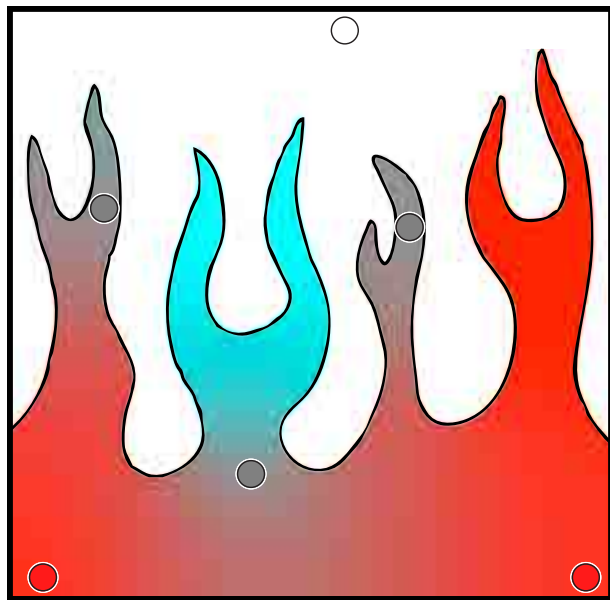
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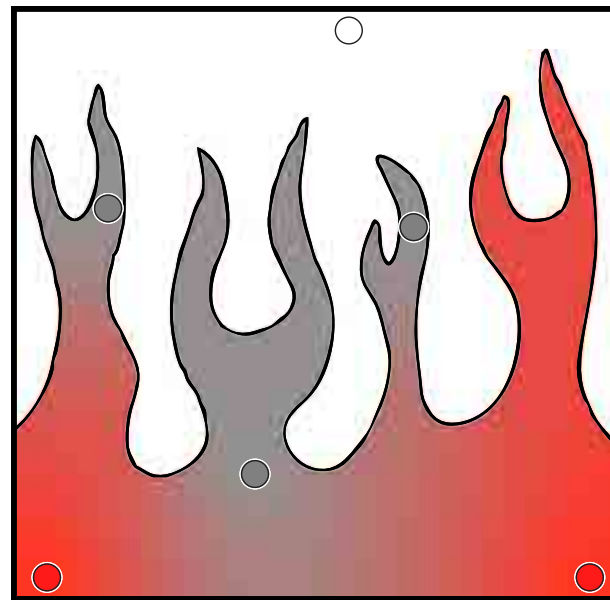
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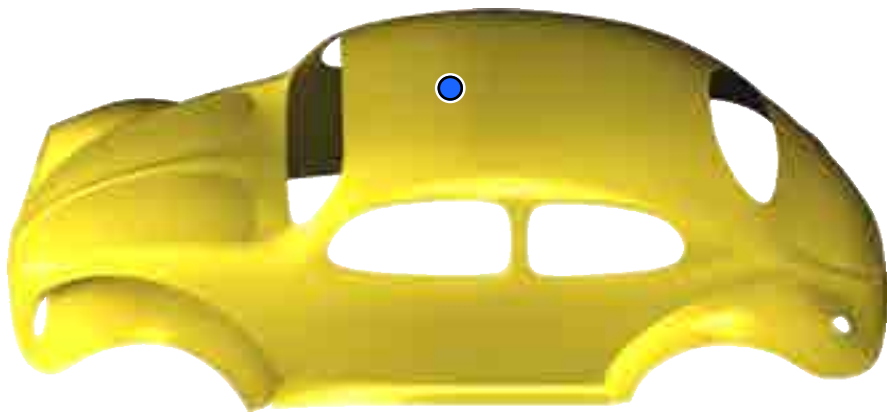


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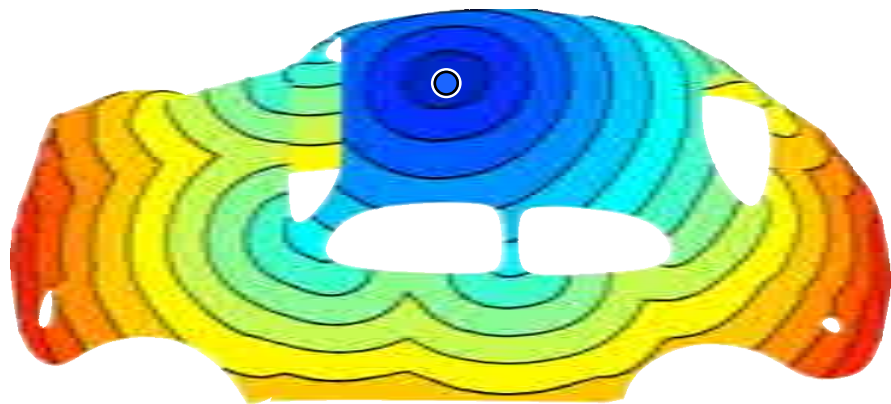
Our Δ^2



Want same control when smoothing data

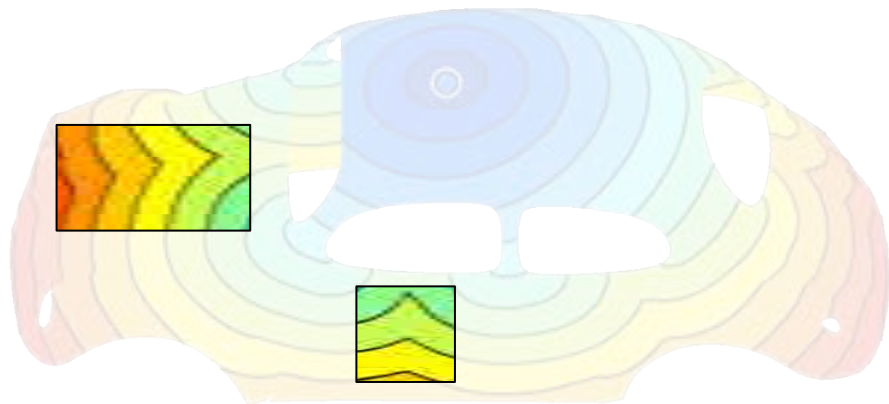


Want same control when smoothing data



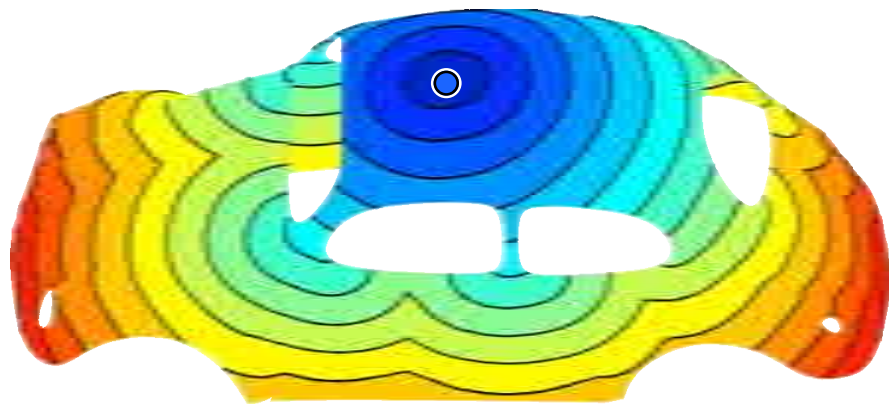
Exact, but sharp geodesic

Want same control when smoothing data

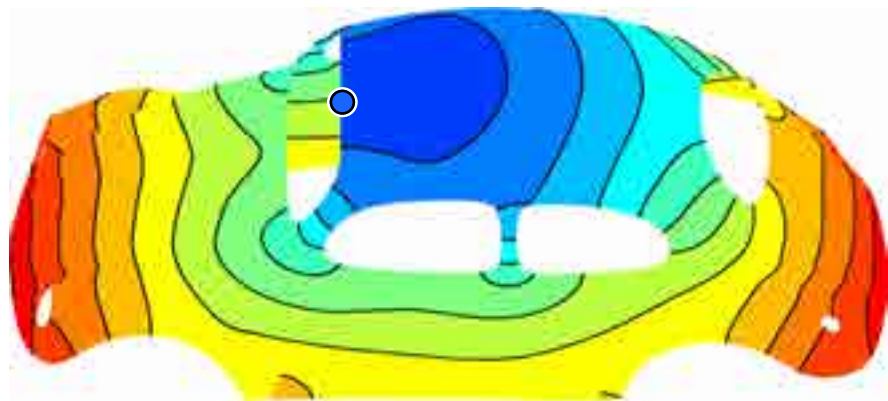


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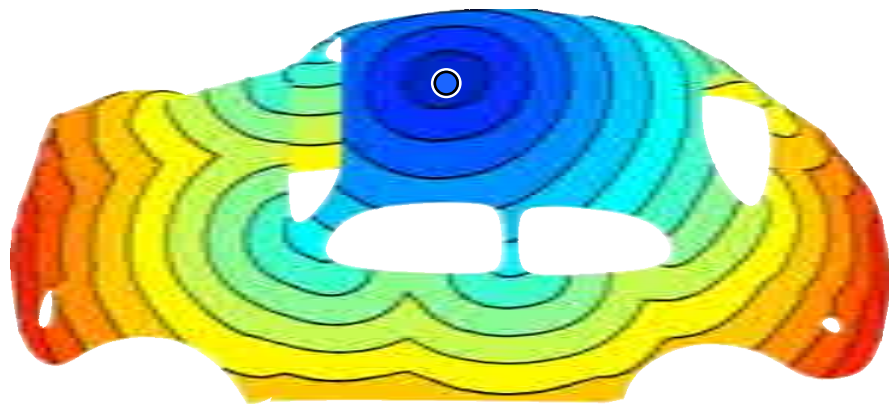


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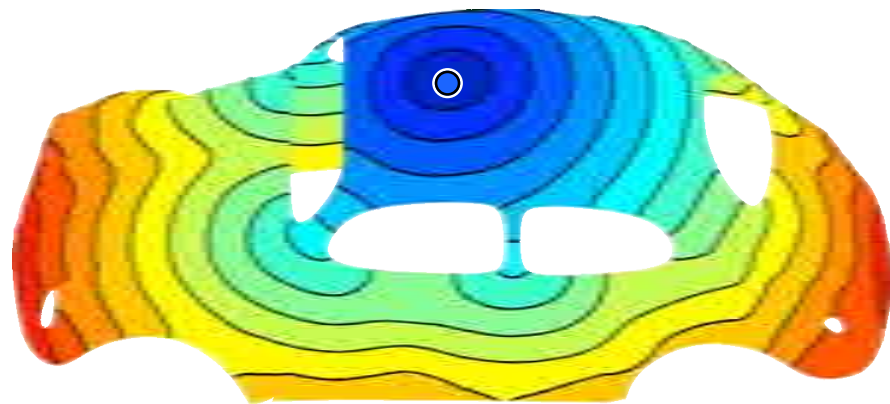


Smooth, but extrema are lost

Want same control when smoothing data



Exact, but sharp geodesic



Smooth *and* maintain extrema

Ideal discrete problem is intractable

$$\arg \min_f E(f)$$

Interpolation functions:

$$E_L(f) = \int_{\mathcal{M}} \|\nabla^k f\|^2 dV, \quad k = 2, 3, \dots$$

Ideal discrete problem is intractable

$$\arg \min_f E(f)$$

Data smoothing:

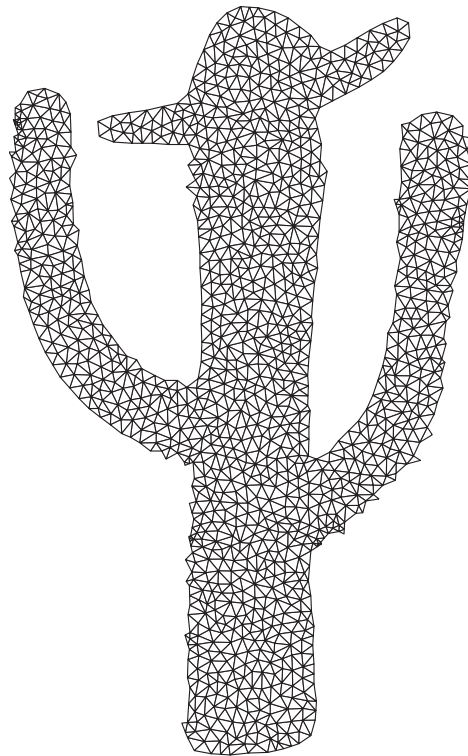
$$E_L(f) = \int_{\mathcal{M}} \|\nabla^k f\|^2 dV, \quad k = 2, 3, \dots$$

$$E_D(f) = \sum_{i \in \mathcal{M}} \|h_i - f_i\|^2$$

$$E(f) = \gamma_L E_L(f) + \gamma_D E_D(f)$$

Ideal discrete problem is intractable

$$\arg \min_f E(f)$$

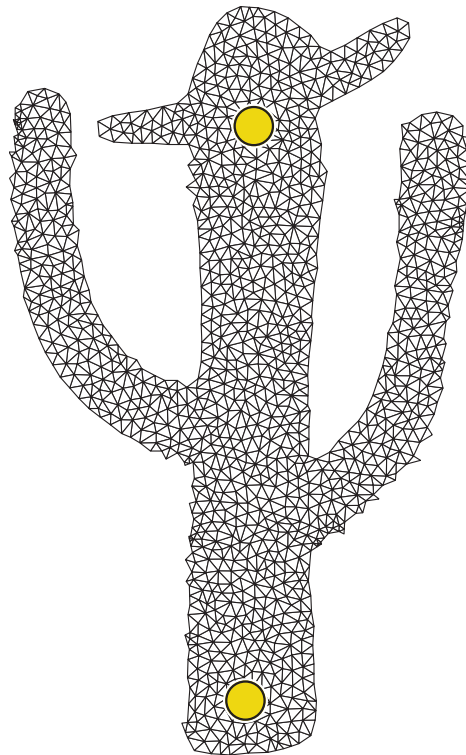


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$$\arg \min_f E(f)$$

$$\text{s.t. } f_{\max} = \textit{known}$$

$$f_{\min} = \textit{known}$$



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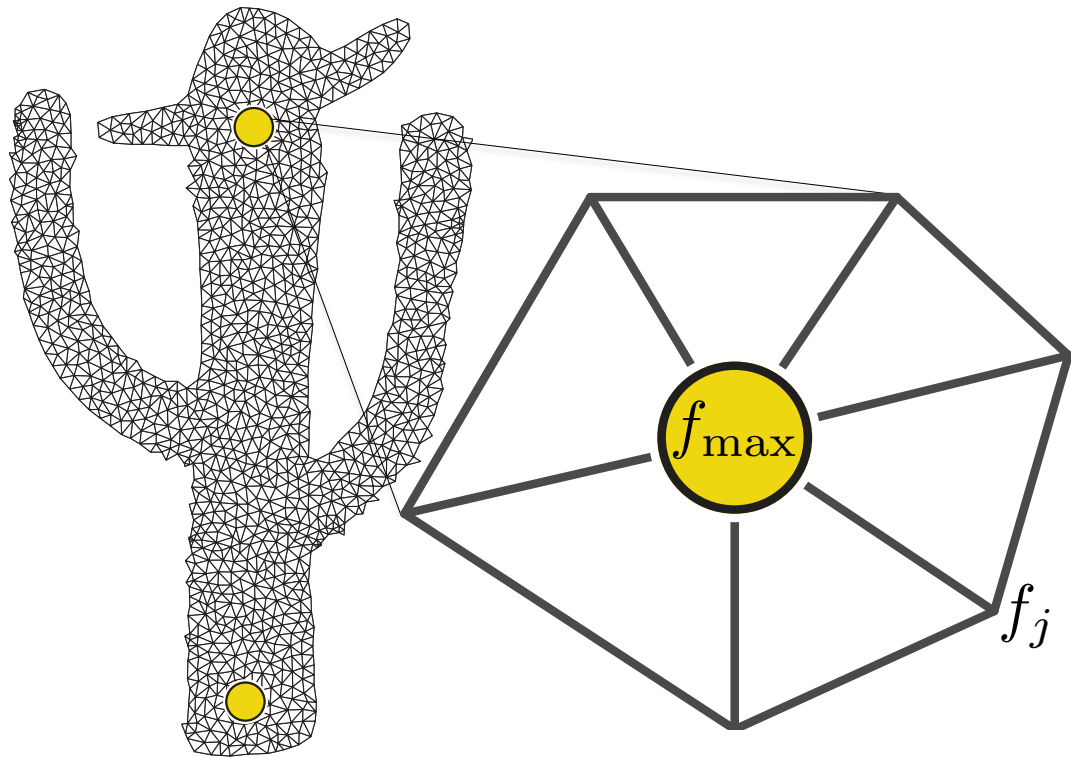
$$\text{s.t. } f_{\max} = \textit{known}$$

$$f_{\min} = \textit{known}$$

linear

$$f_j < f_{\max}$$

$$f_j > f_{\min}$$



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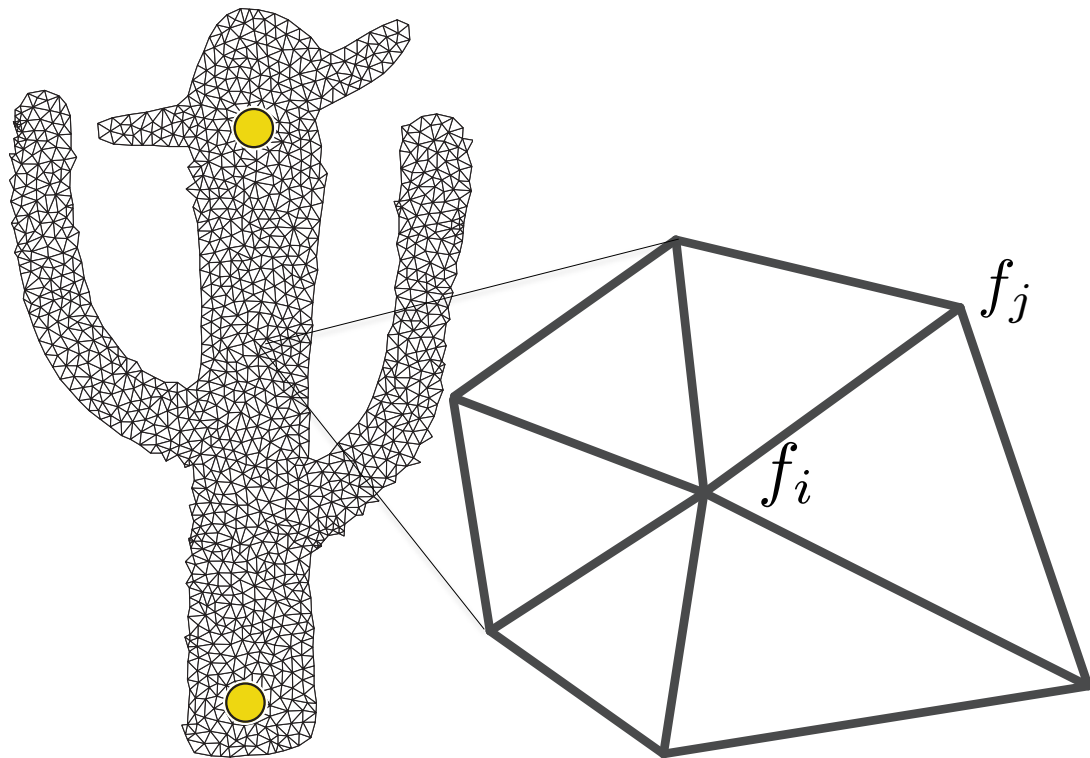
$$f_j < f_{\max}$$

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nonlinear

$$f_i > \min_{j \in \mathcal{N}(i)} f_j$$

$$f_i < \max_{j \in \mathcal{N}(i)} f_j$$



Assume we have a feasible solution

$$\arg \min_f E(f)$$

$$\text{s.t. } f_{\max} = \textit{known}$$

$$f_{\min} = \textit{known}$$

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$$f_j < f_{\max}$$

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“Representative function” u

$$u_j < u_{\max}$$

$$u_j > u_{\min}$$

handles

$$u_i > \min_{j \in \mathcal{N}(i)} u_j$$

$$u_i < \max_{j \in \mathcal{N}(i)} u_j$$

interior

Assume we have a feasible solution

“Representative function” u

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interior

$$u_i > \min_{j \in \mathcal{N}(i)} u_j$$

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Copy “monotonicity” of representative

$$\arg \min_f E(f)$$

$$\text{s.t. } f_{\max} = \textit{known}$$

$$f_{\min} = \textit{known}$$

$$(f_i - f_j)(u_i - u_j) > 0 \quad \text{linear} \quad \forall (i, j) \in \mathcal{E}$$

At least one edge in either
direction per vertex

Rewrite as conic optimization

QP

$$\begin{aligned} & \underset{\mathbf{f}}{\text{minimize}} && \frac{1}{2} \|\mathbf{F} \mathbf{f}\|^2 + \mathbf{c}^\top \mathbf{f} + \text{const} \\ & \text{subject to} && \mathbf{A}_{leq}^\top \mathbf{f} \leq \mathbf{b}_{leq}, \\ & && \mathbf{f} \leq \mathbf{u}_f, \quad \mathbf{f} \geq \mathbf{l}_f \end{aligned}$$



Conic

$$\begin{aligned} & \underset{\begin{bmatrix} \mathbf{f} \\ \mathbf{t} \\ v \end{bmatrix}}{\text{minimize}} && \begin{bmatrix} \mathbf{c}^\top & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{f} \\ \mathbf{t} \\ v \end{bmatrix} + \text{const} \\ & \text{subject to} && \begin{bmatrix} \mathbf{F} & -\mathbf{I} & 0 \\ \mathbf{A}_{leq}^\top & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{f} \\ \mathbf{t} \\ v \end{bmatrix} \geq \begin{bmatrix} 0 \\ -\infty \end{bmatrix} \\ & && \begin{bmatrix} \mathbf{F} & -\mathbf{I} & 0 \\ \mathbf{A}_{leq}^\top & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{f} \\ \mathbf{t} \\ v \end{bmatrix} \leq \begin{bmatrix} 0 \\ \mathbf{b}_{leq} \end{bmatrix} \\ & && \begin{bmatrix} \mathbf{f} \\ \mathbf{t} \\ v \end{bmatrix} \leq \begin{bmatrix} \mathbf{u}_f \\ \infty \\ \infty \end{bmatrix} \\ & && \begin{bmatrix} \mathbf{f} \\ \mathbf{t} \\ v \end{bmatrix} \geq \begin{bmatrix} \mathbf{l}_f \\ -\infty \\ 0 \end{bmatrix} \\ & && 2v \geq \sum_i t_i^2 \end{aligned}$$

Optimize with MOSEK

We always have harmonic representative

$$\arg \min_u \frac{1}{2} \int_{\Omega} \|\nabla u\|^2 dV$$

We always have harmonic representative

$$\begin{aligned} \arg \min_u \quad & \frac{1}{2} \int_{\Omega} \|\nabla u\|^2 dV \\ \text{s.t.} \quad & u_{\max} = 1 \end{aligned}$$

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$$\text{s.t.} \quad u_{\max} = 1$$

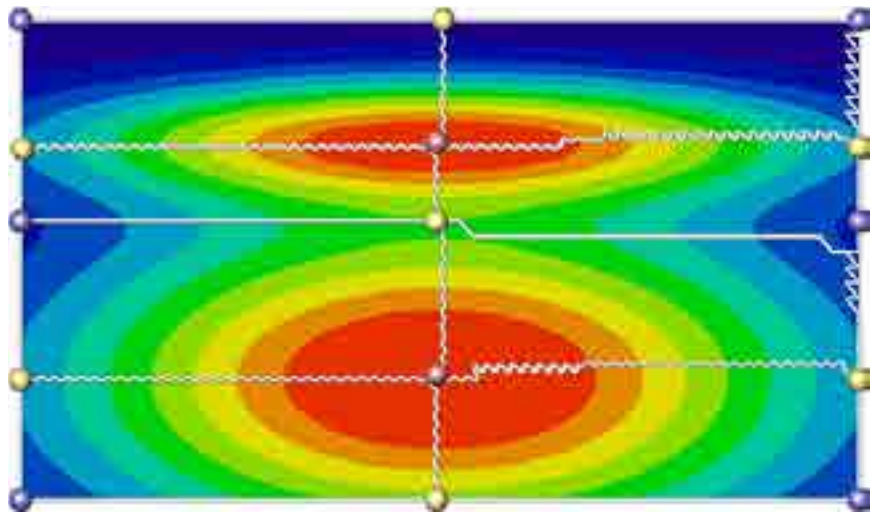
$$\text{s.t.} \quad u_{\min} = 0$$

We always have harmonic representative

$$\begin{aligned} \arg \min_u \quad & \frac{1}{2} \int_{\Omega} \|\nabla u\|^2 dV \\ \text{s.t.} \quad & u_{\max} = 1 \\ \text{s.t.} \quad & u_{\min} = 0 \end{aligned}$$

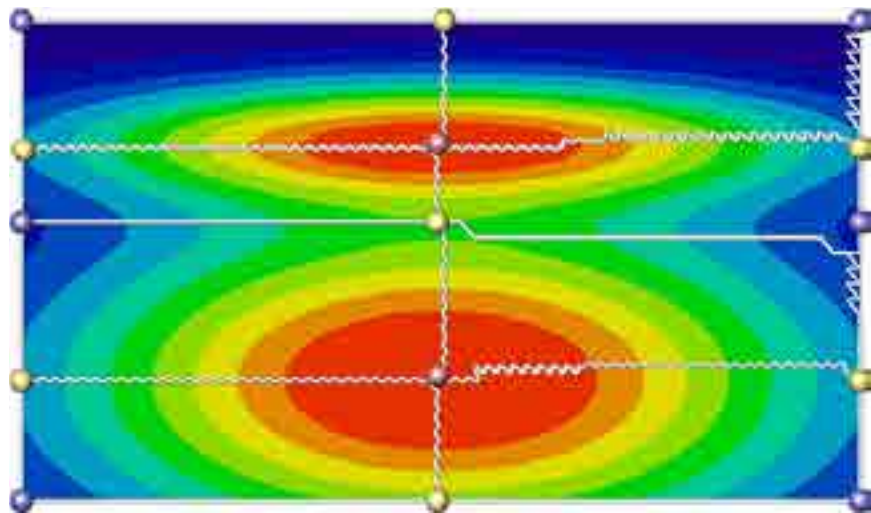
Works well when no input function exists

Data energy may fight harmonic representative

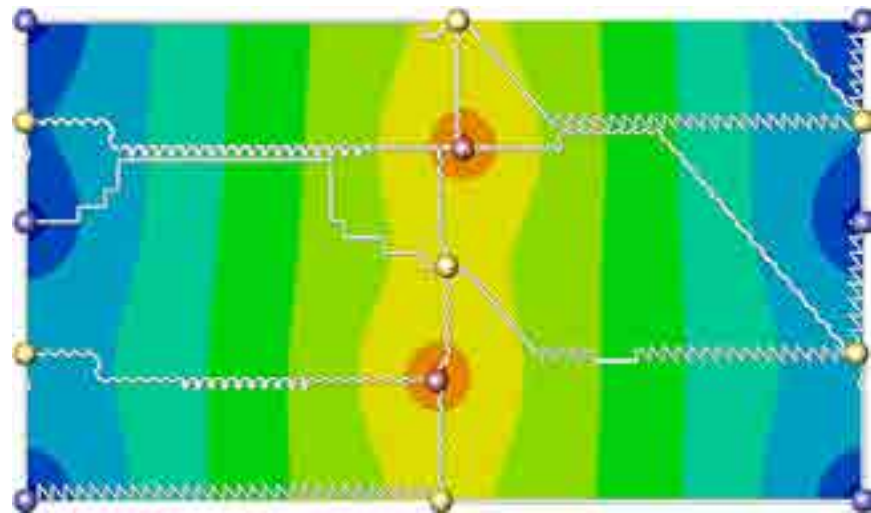


Anisotropic input data

Data energy may fight harmonic representative

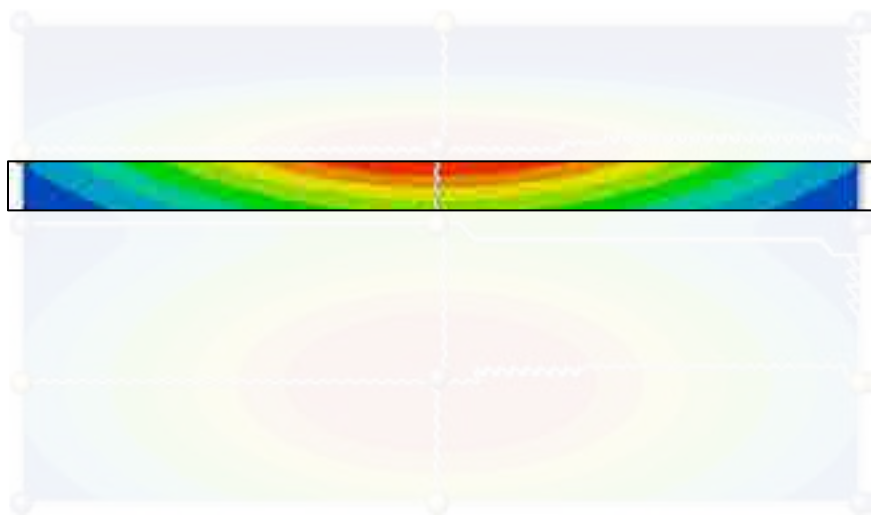


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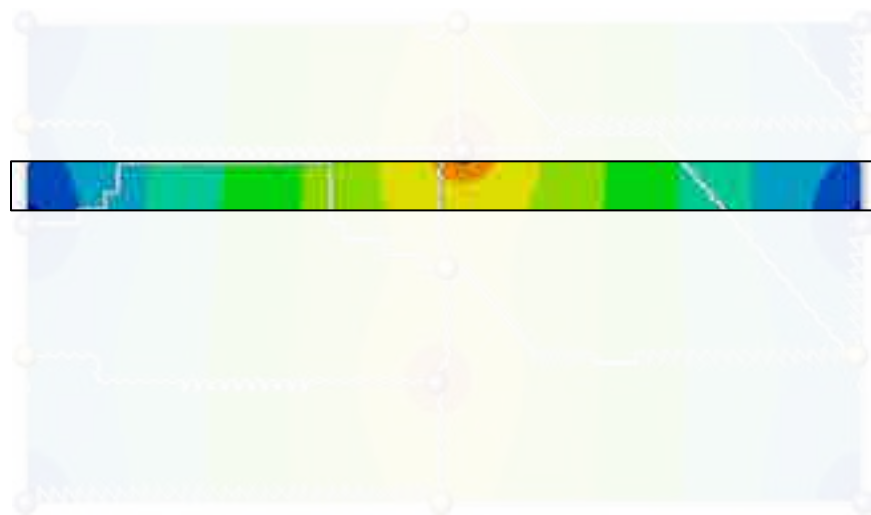


Harmonic representative

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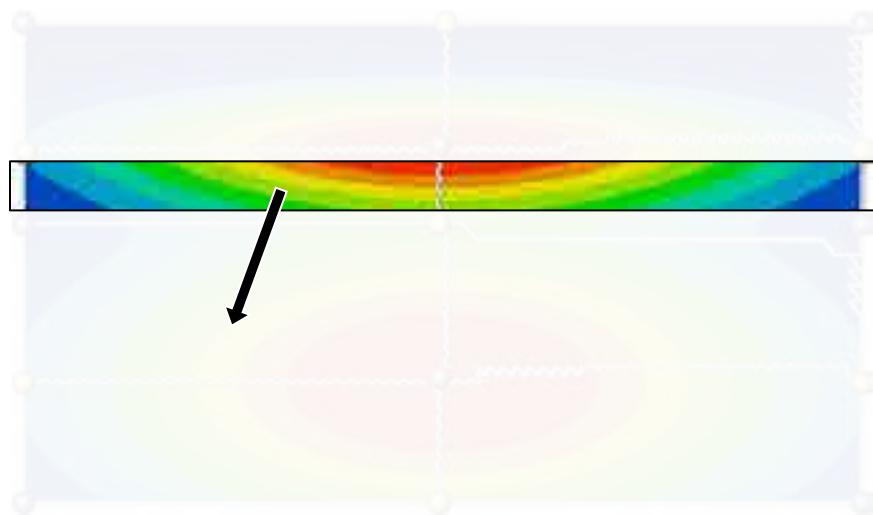


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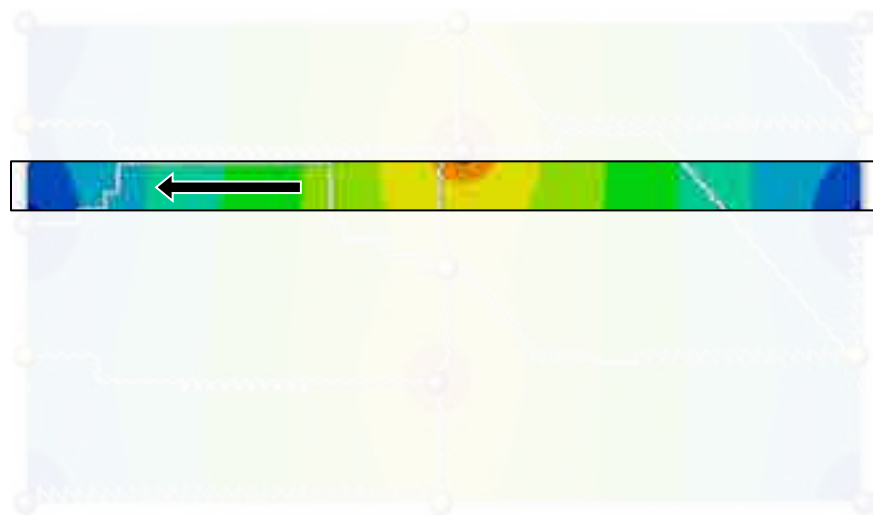


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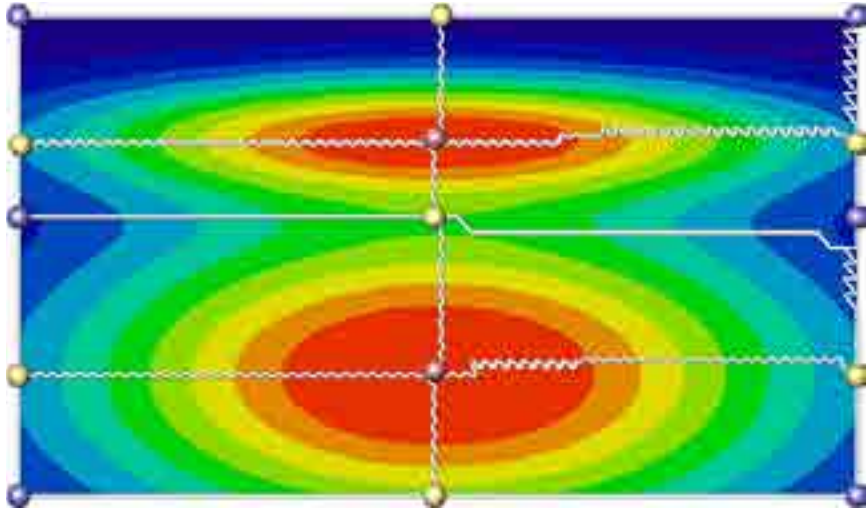


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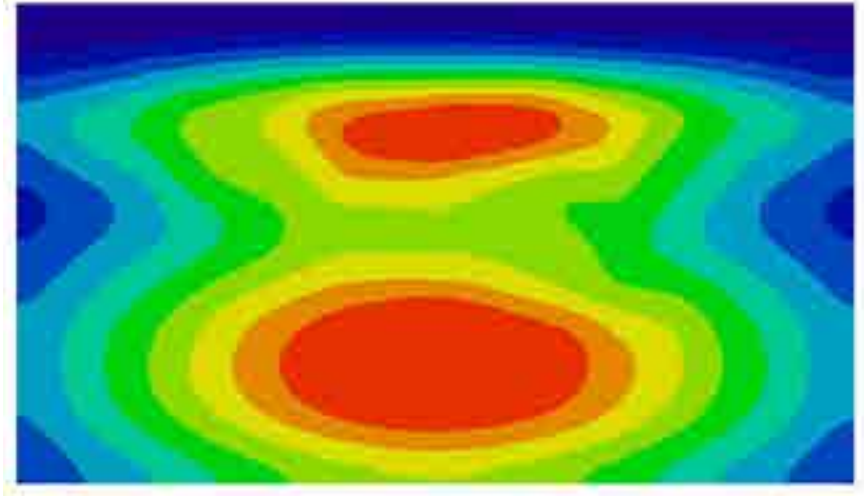


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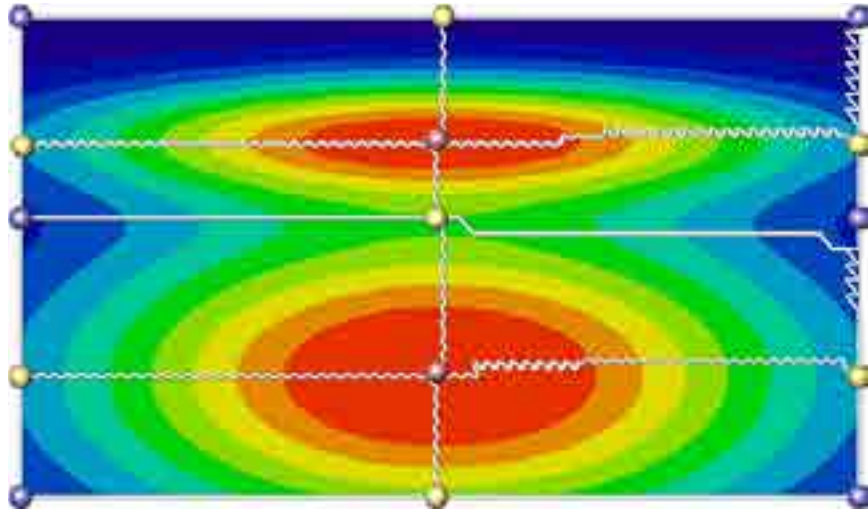


Anisotropic input data

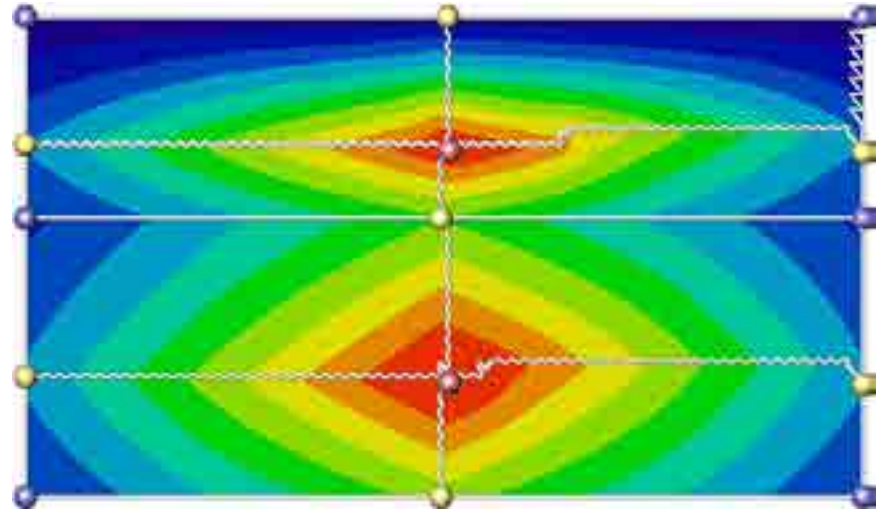


Resulting solution with large γ_D

If data exists, copy topology, too

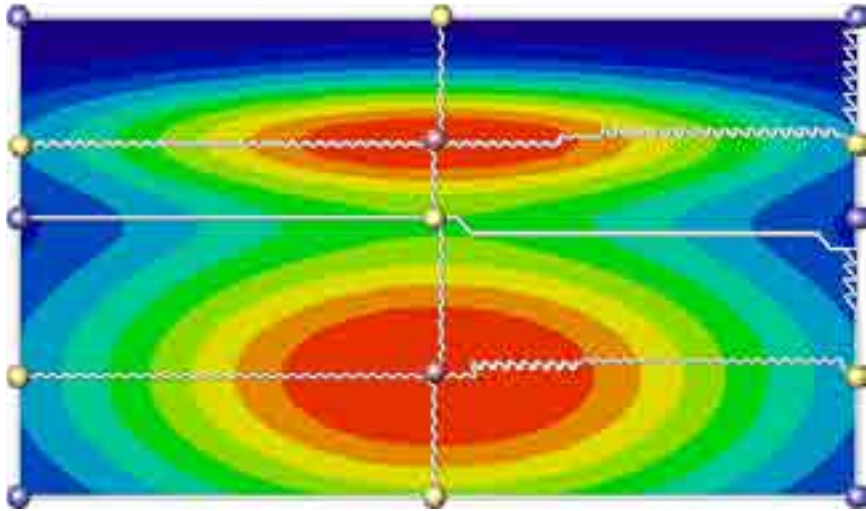


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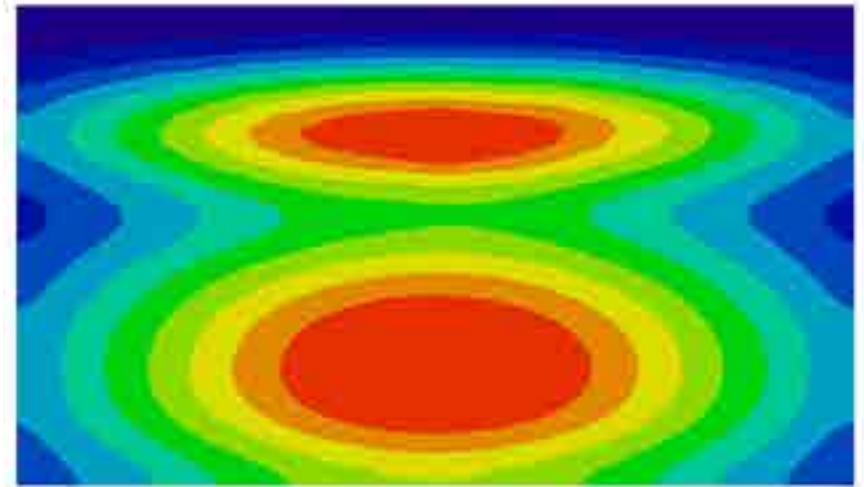


[Weinkauff et al. 2010]
representative

If data exists, copy topology, too



Anisotropic input data



Resulting solution with large γ_D

Final algorithm is simple and efficient

- *Data smoothing*: topology-aware representative
 - Morse-smale + linear solve ~milliseconds

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- Conic optimization
 - 2D ~milliseconds, 3D ~seconds

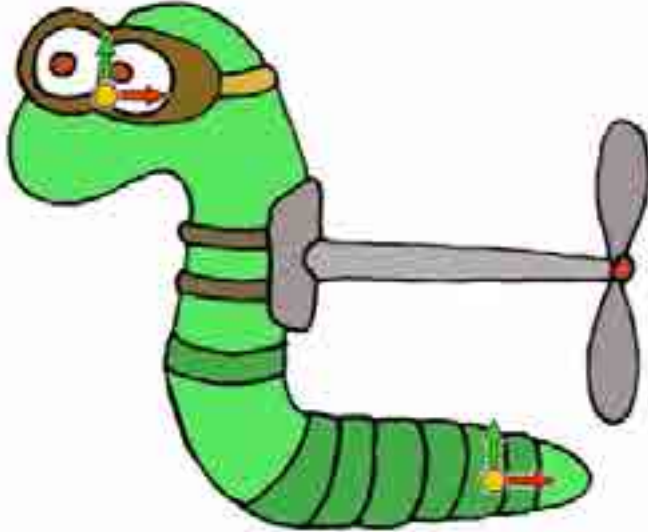
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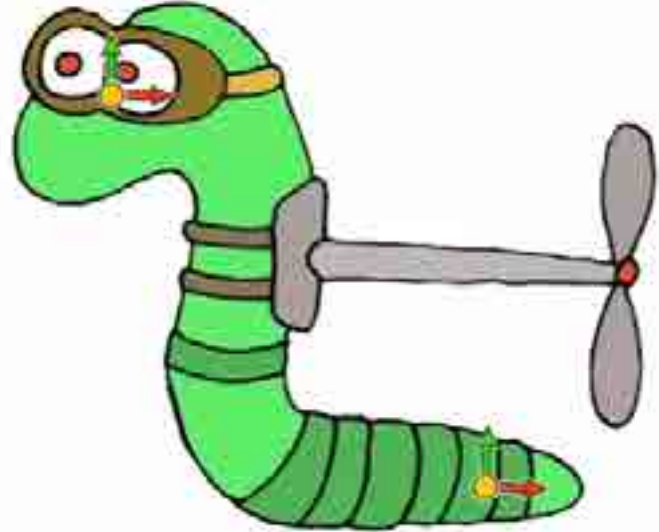
Interpolation: functions are precomputed

We preserve troublesome appendages

Bounded Δ^2

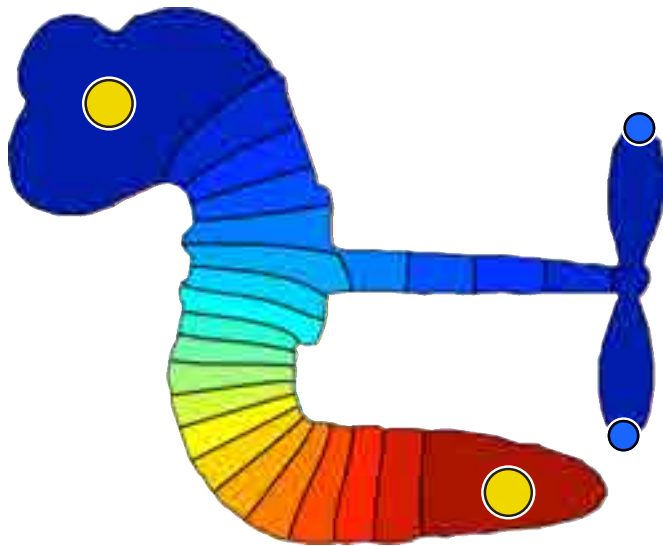


Our Δ^2

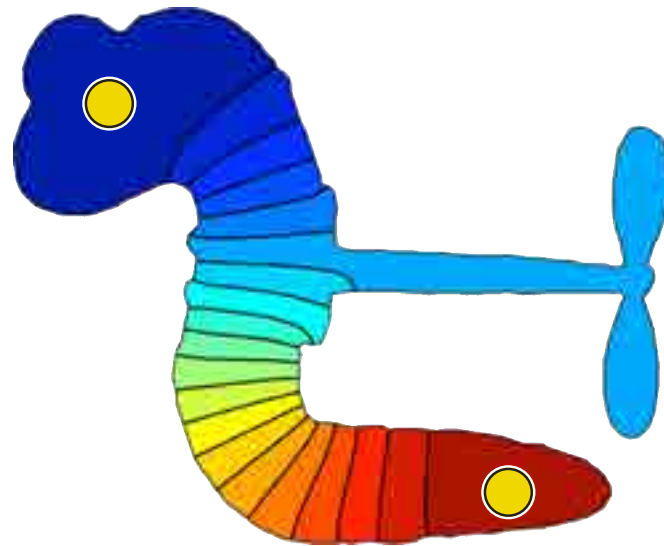


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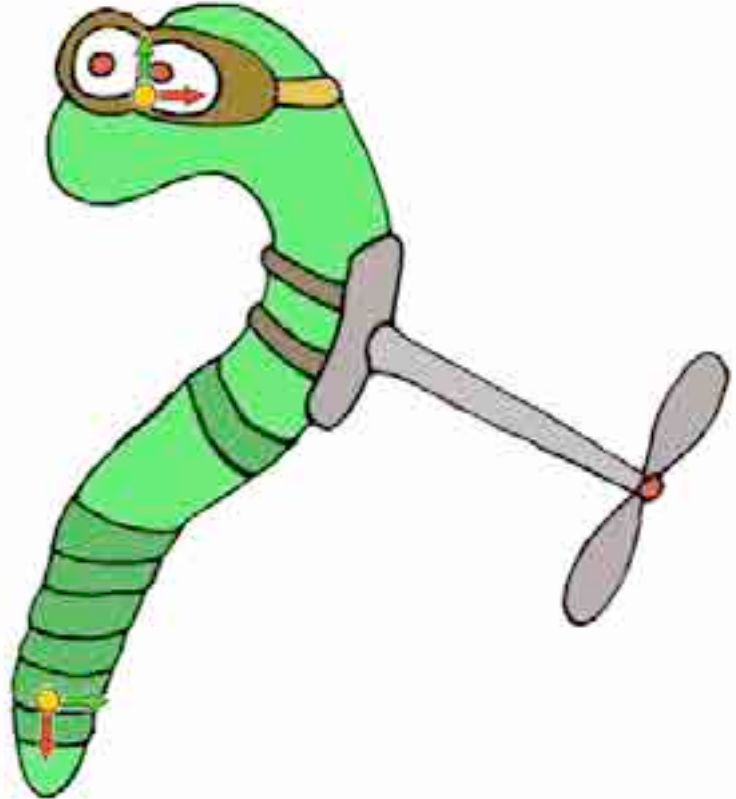


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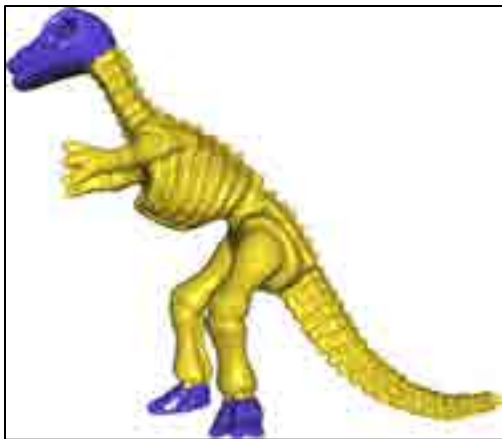
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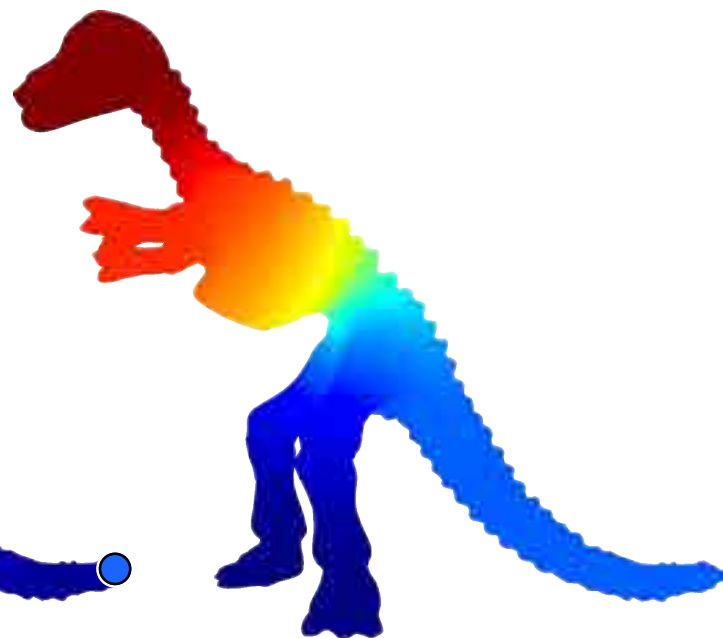
Our Δ^2



Our weights attach appendages to body



[Botsch & Kobbelt 2004,
Jacobson et al. 2011]



Our method

Extrema glue appendages to far-away handles



[Botsch & Kobbelt 2004, Jacobson et al. 2011]

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Our method

Our weights attach appendages to body



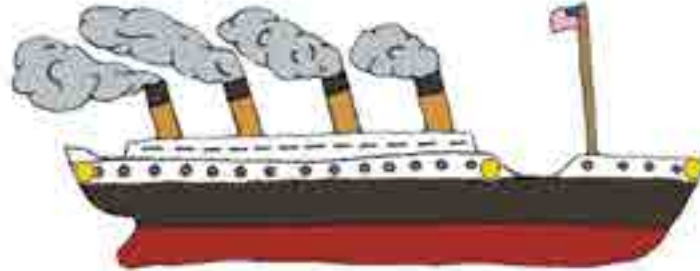
Our method

Extrema distort small features

Unconstrained Δ^2 [Botsch & Kobbelt 2004]



weight of middle point

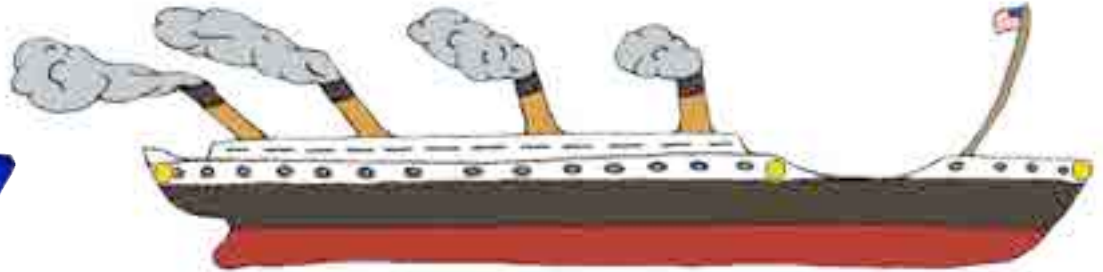


Extrema distort small features

Unconstrained Δ^2 [Botsch & Kobbelt 2004]

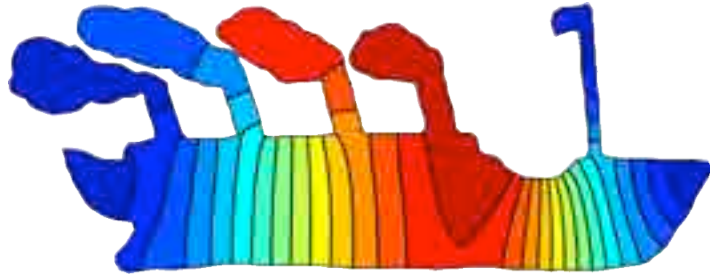


weight of middle point

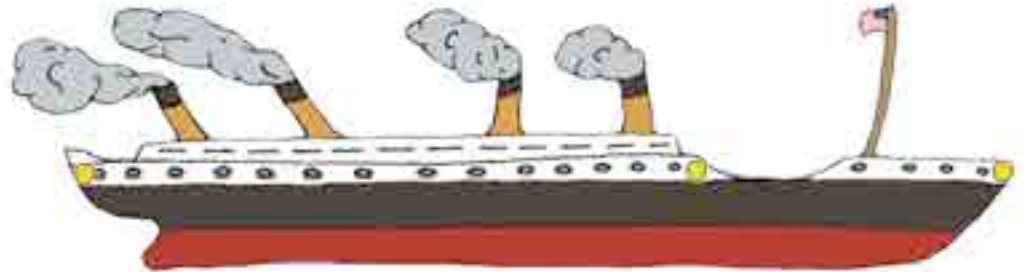


Extrema distort small features

Bounded Δ^2 [Jacobson et al. 2011]

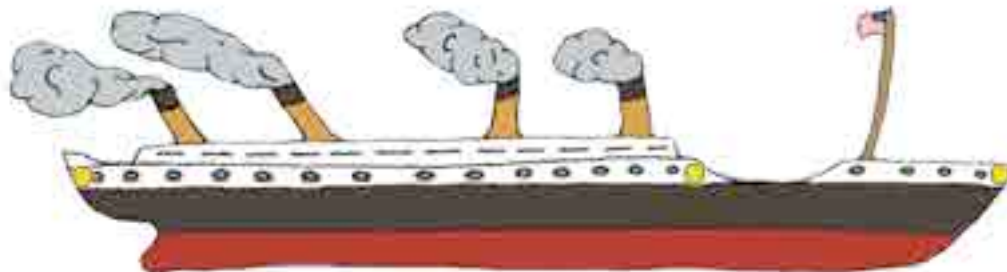
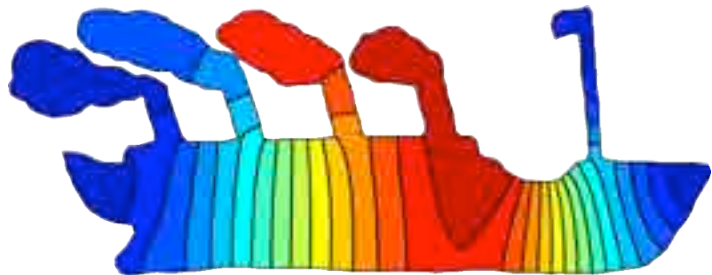


weight of middle point

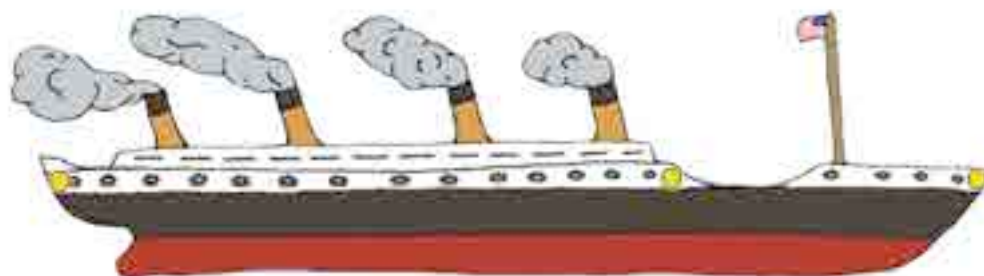


“Monotonicity” helps preserve small features

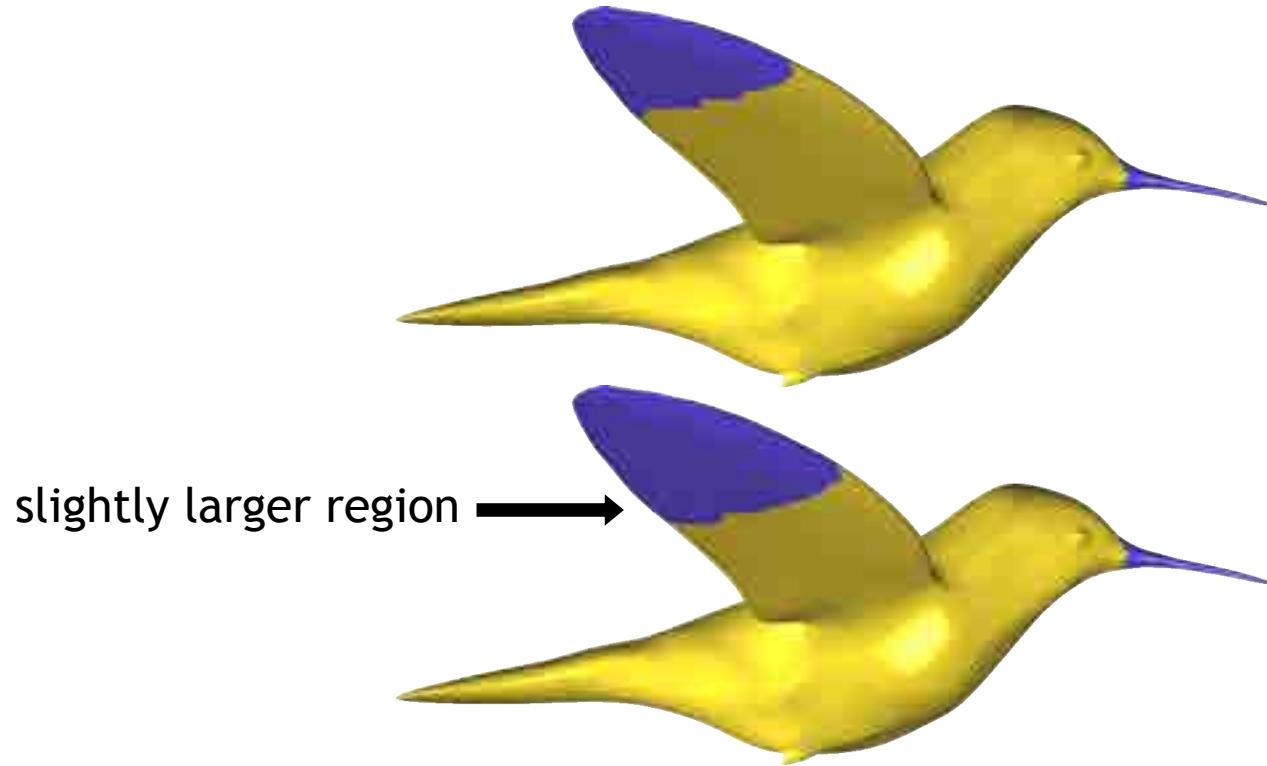
Bounded Δ^2 [Jacobson et al. 2011]



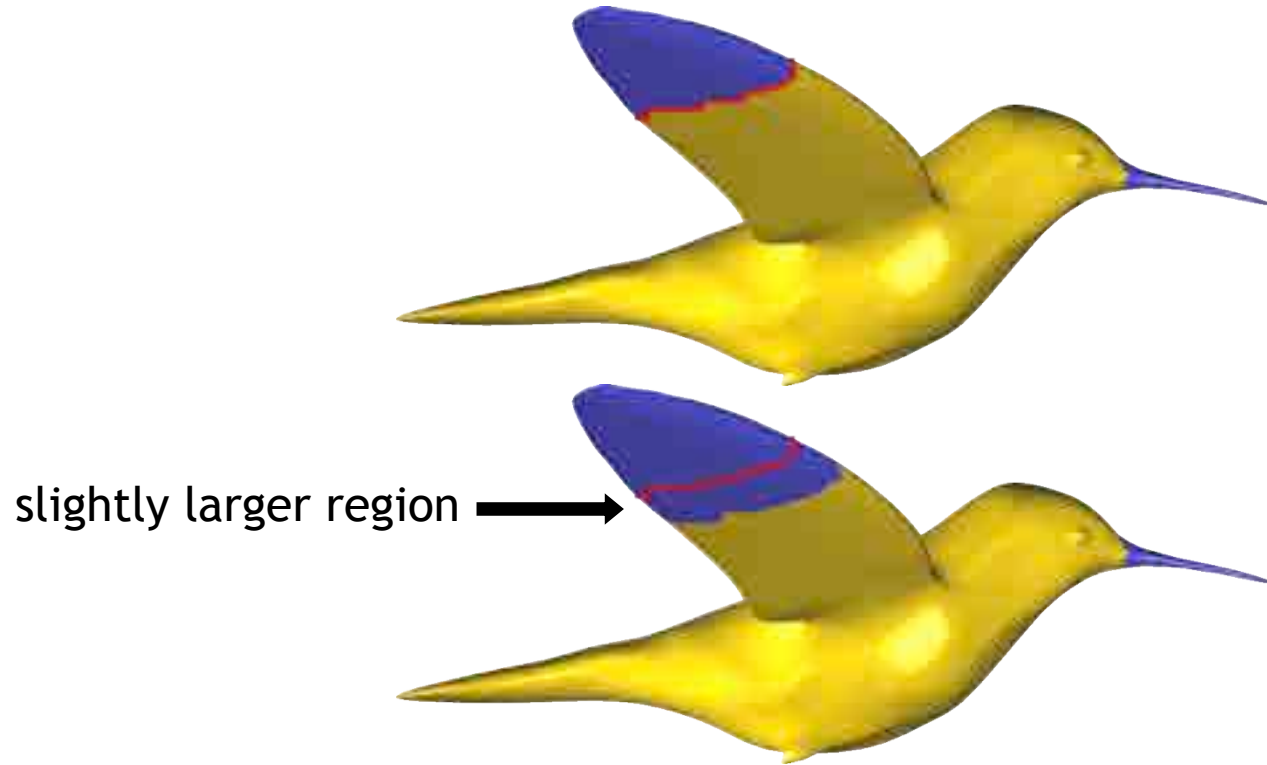
Our Δ^2



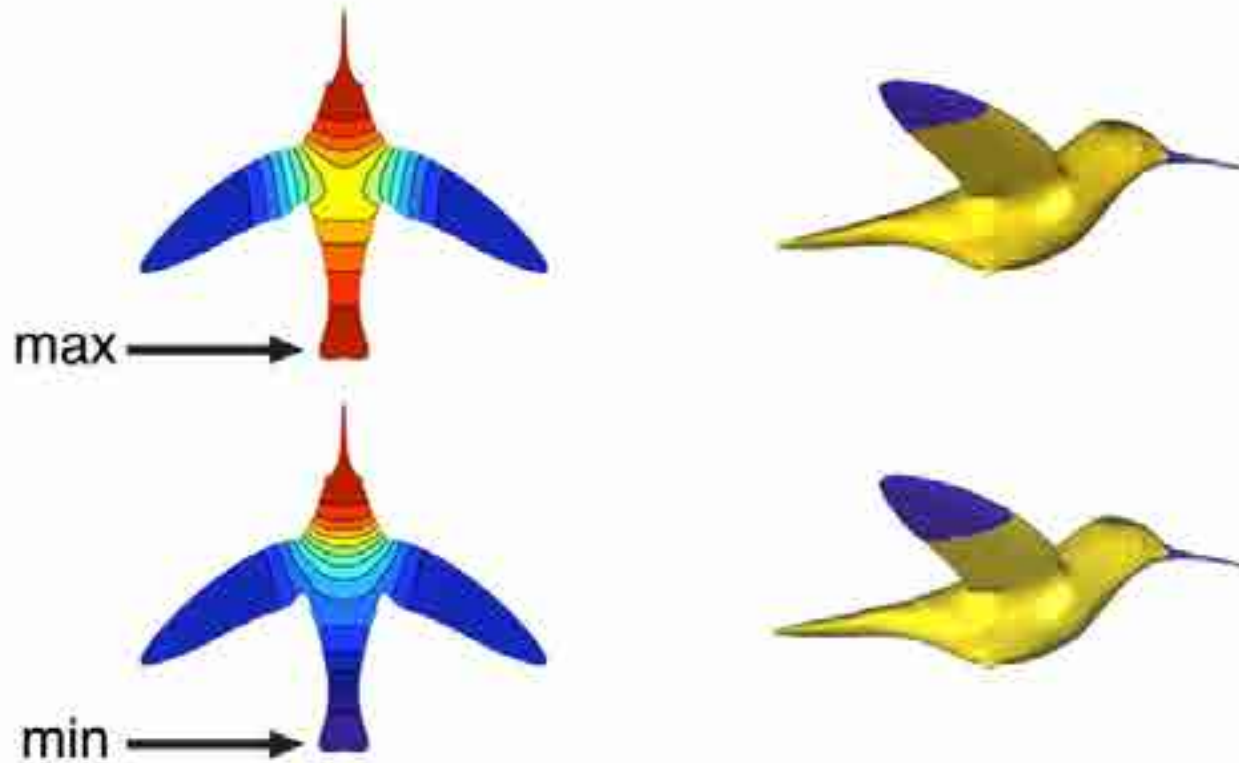
Spurious extrema are unstable, may “flip”



Spurious extrema are unstable, may “flip”

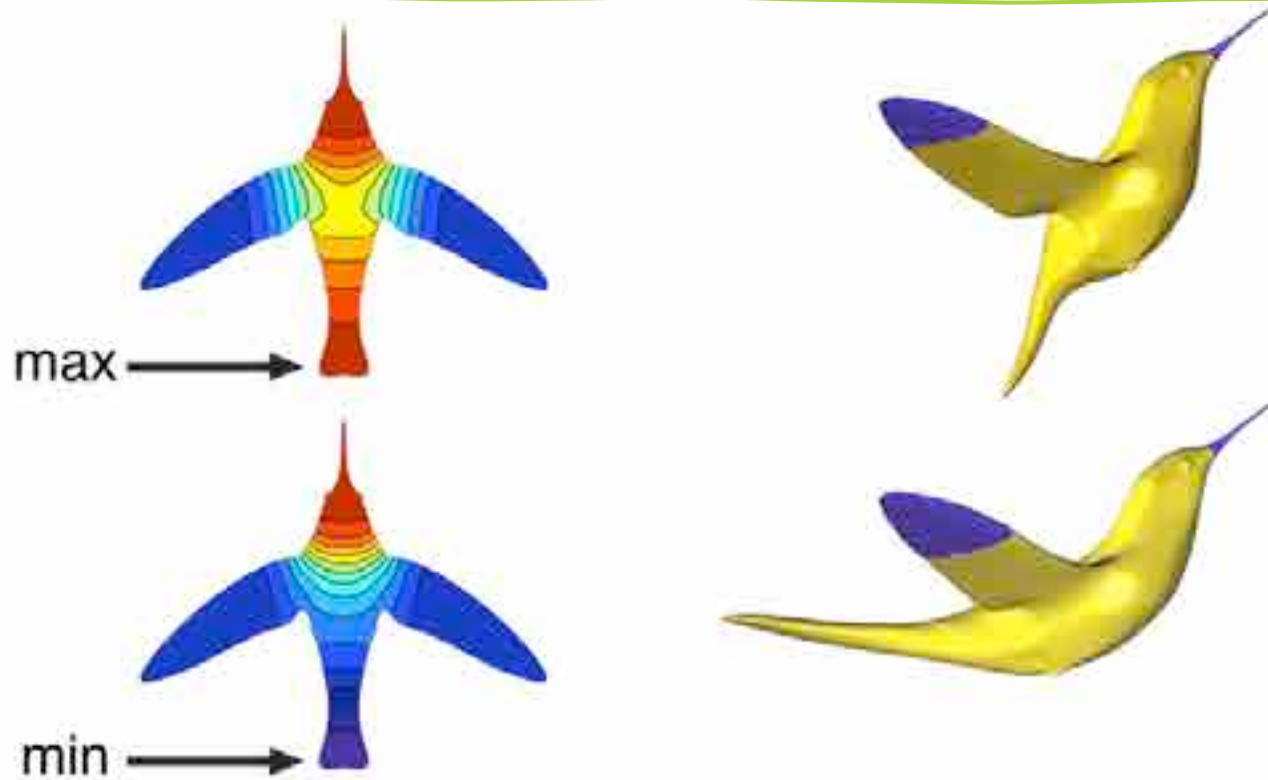


Spurious extrema are unstable, may “flip”



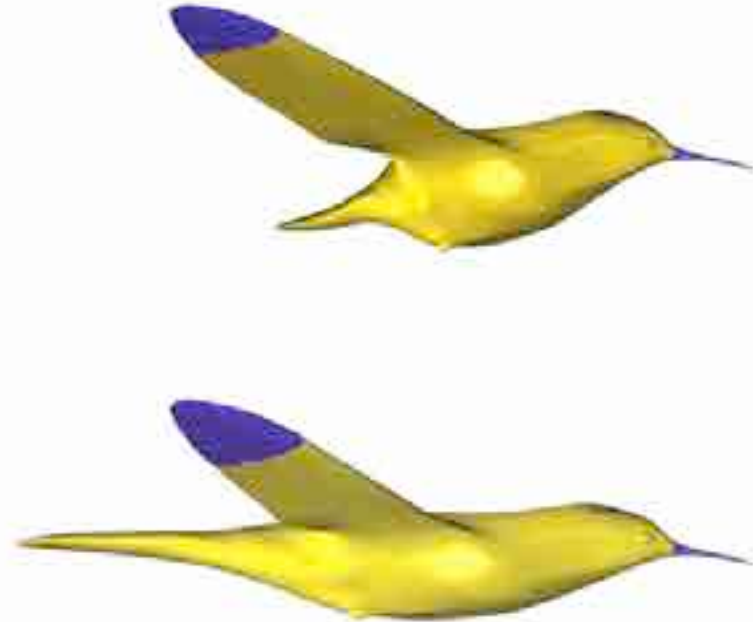
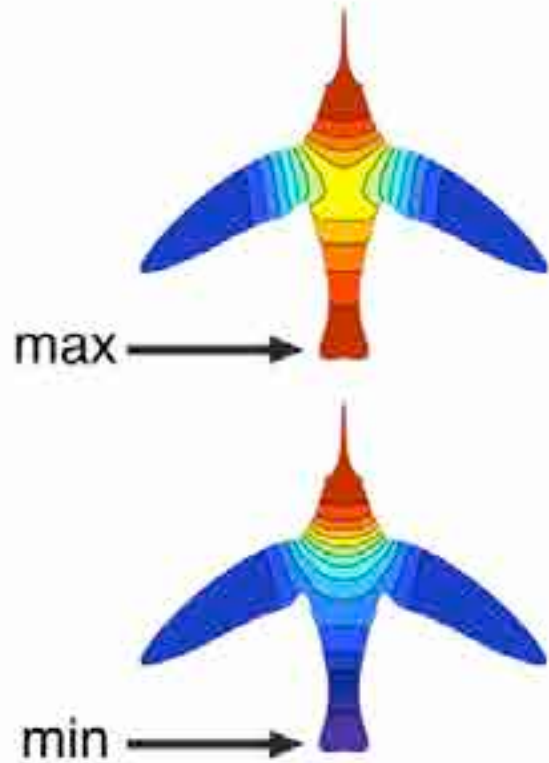
Unconstrained Δ^3 [Botsch & Kobbelt, 2004]

Spurious extrema are unstable, may “flip”



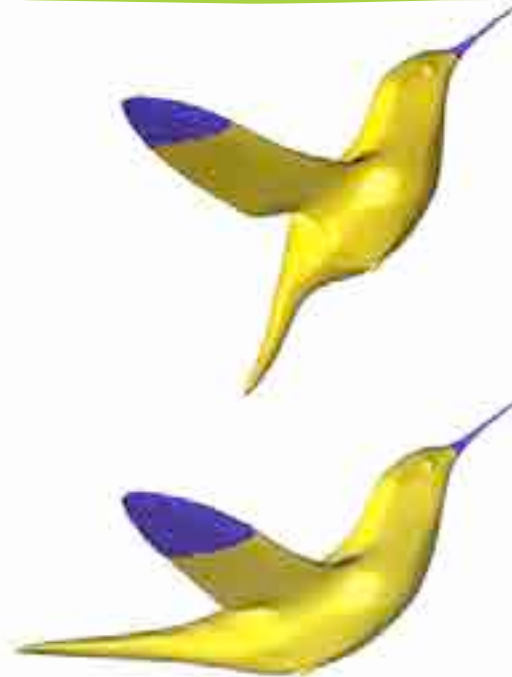
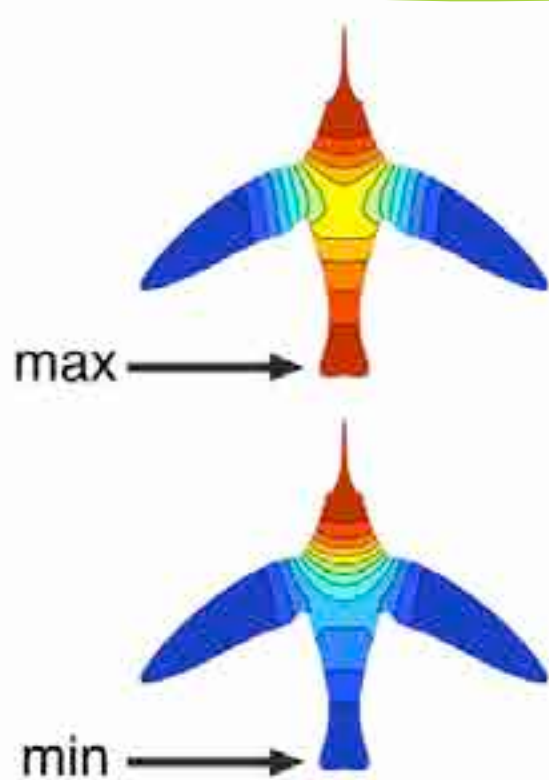
Unconstrained Δ^3 [Botsch & Kobbelt, 2004]

Spurious extrema are unstable, may “flip”



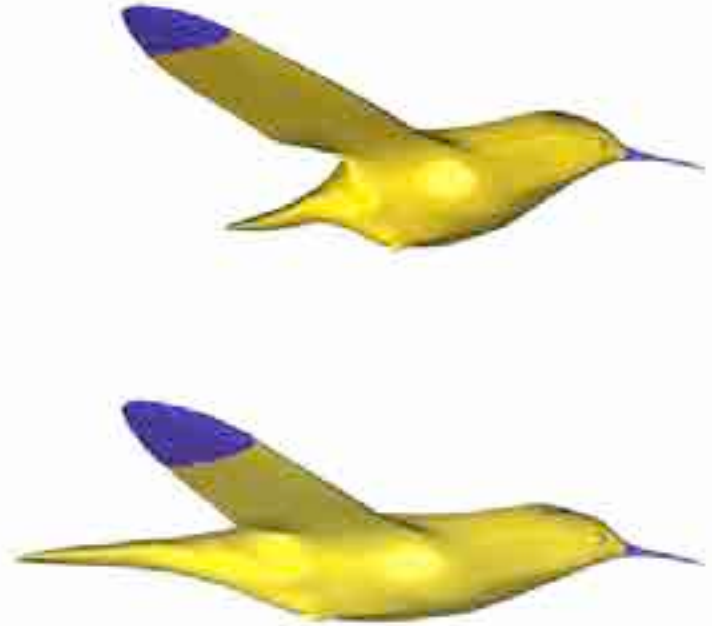
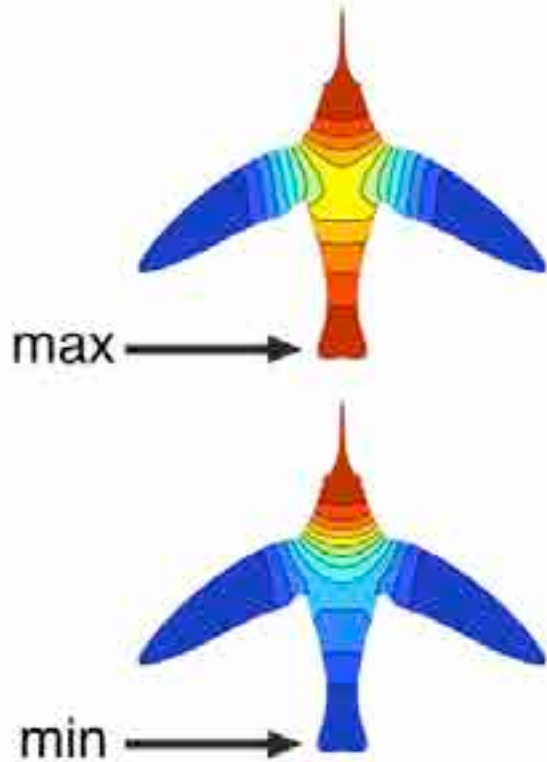
Unconstrained Δ^3 [Botsch & Kobbelt, 2004]

Spurious extrema are unstable, may “flip”



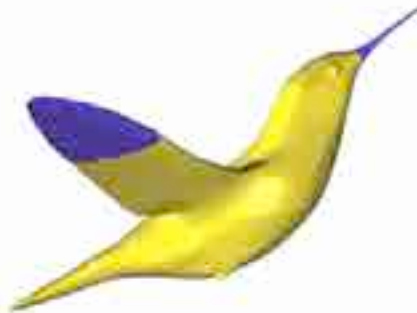
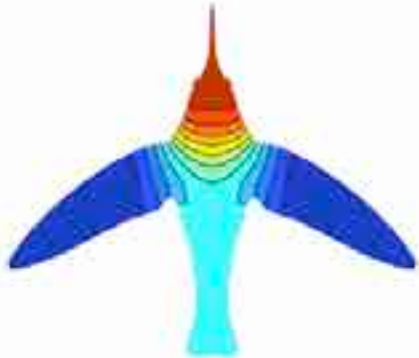
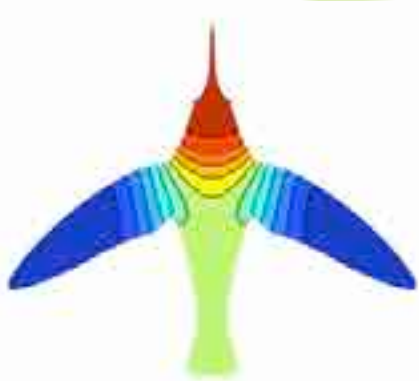
Bounded Δ^3

Spurious extrema are unstable, may “flip”



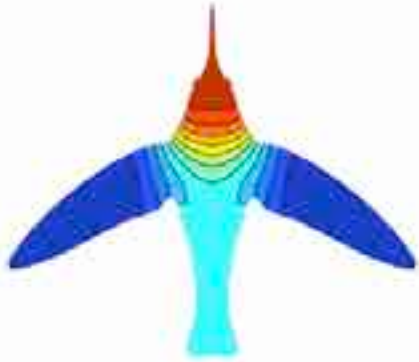
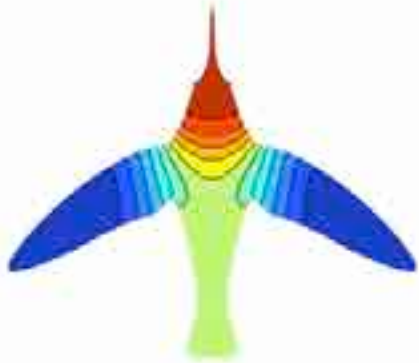
Bounded Δ^3

Lack of extrema leads to more stability



Our Δ^3

Lack of extrema leads to more stability



Our Δ^3

Even control continuity at extrema

Original



Even control continuity at extrema

Original



Direct extension of [Botsch & Kobbelt 2004]

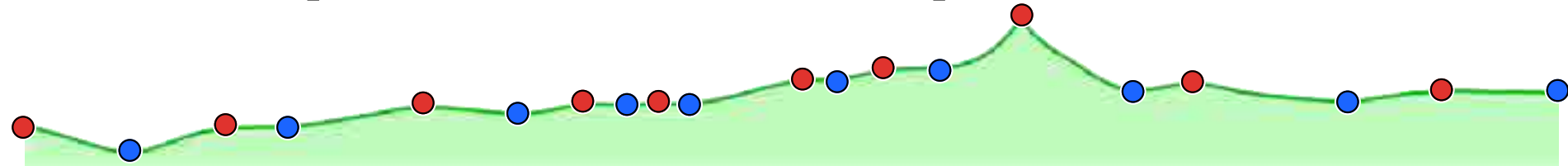


Even control continuity at extrema

Original



[Botsch & Kobbelt 2004] + data term



Even control continuity at extrema

Original



Our method without data term



Even control continuity at extrema

Original

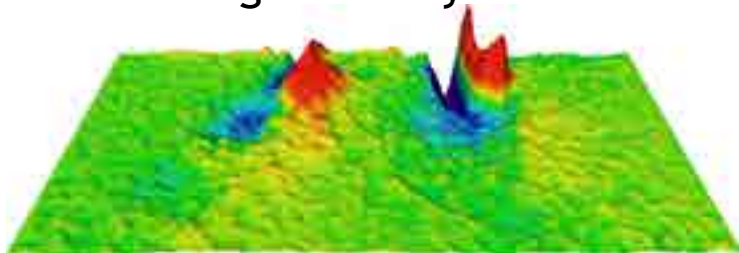


Our method with data term



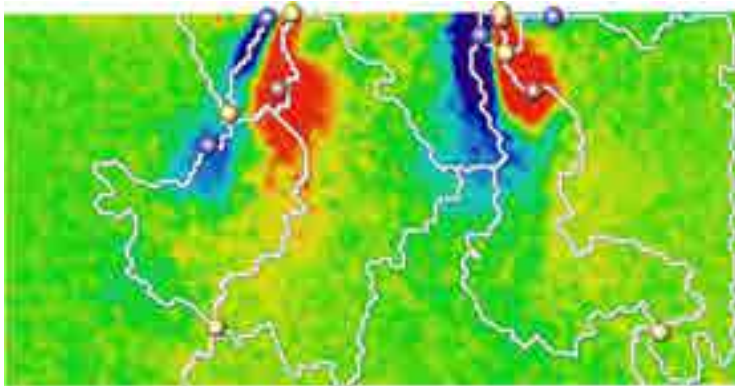
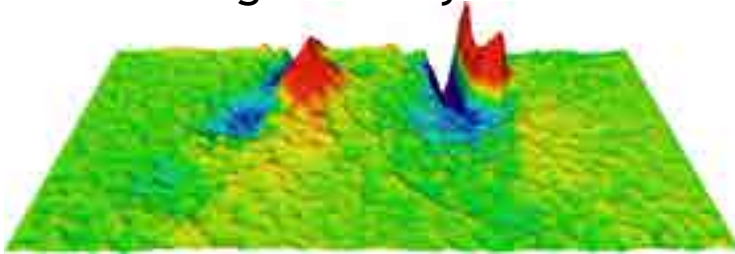
Reproduces results of Weinkauff et al. 2010...

Original noisy data



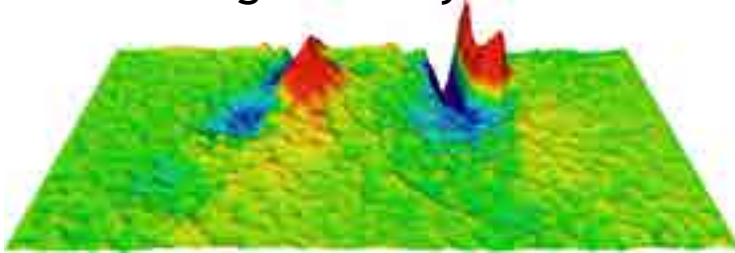
Reproduces results of Weinkauff et al. 2010...

Original noisy data

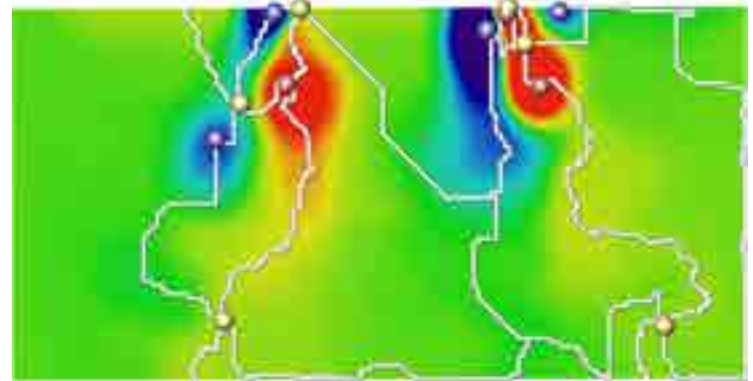
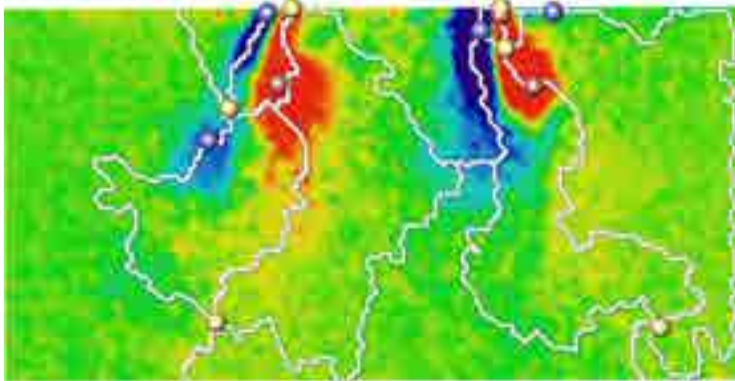
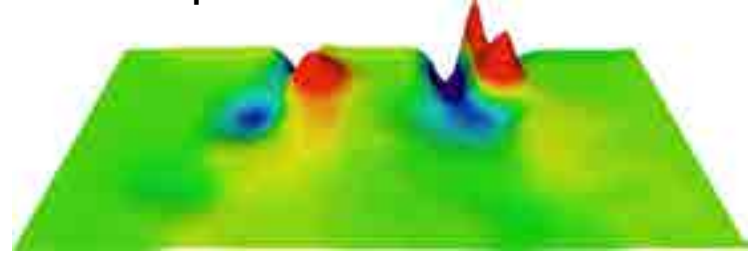


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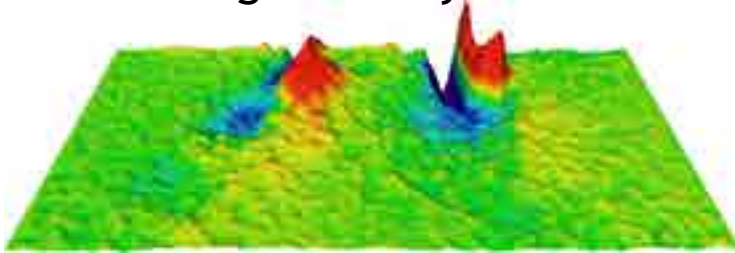


Simplified and smoothed

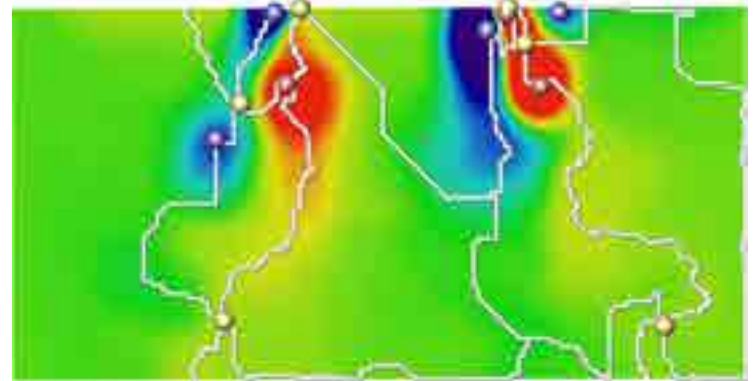
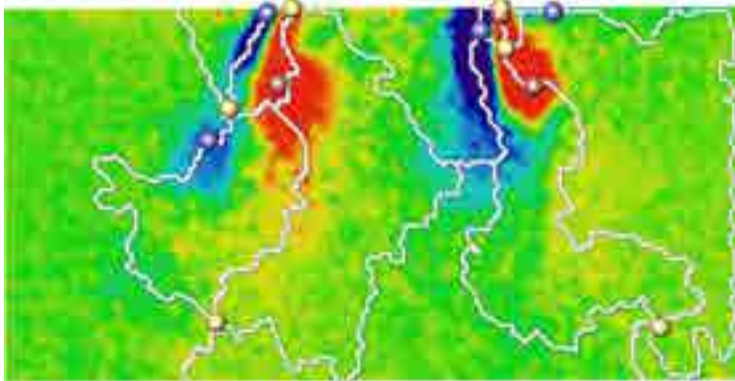
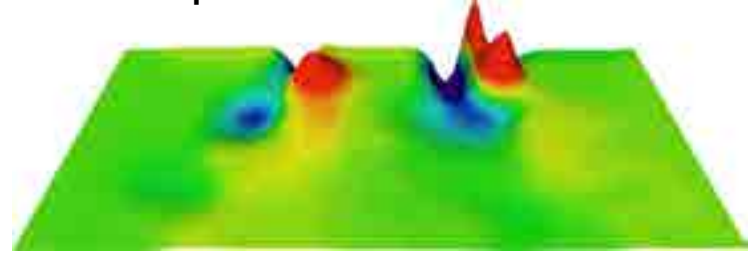


Reproduces results of Weinkauff et al. 2010...

Original noisy data

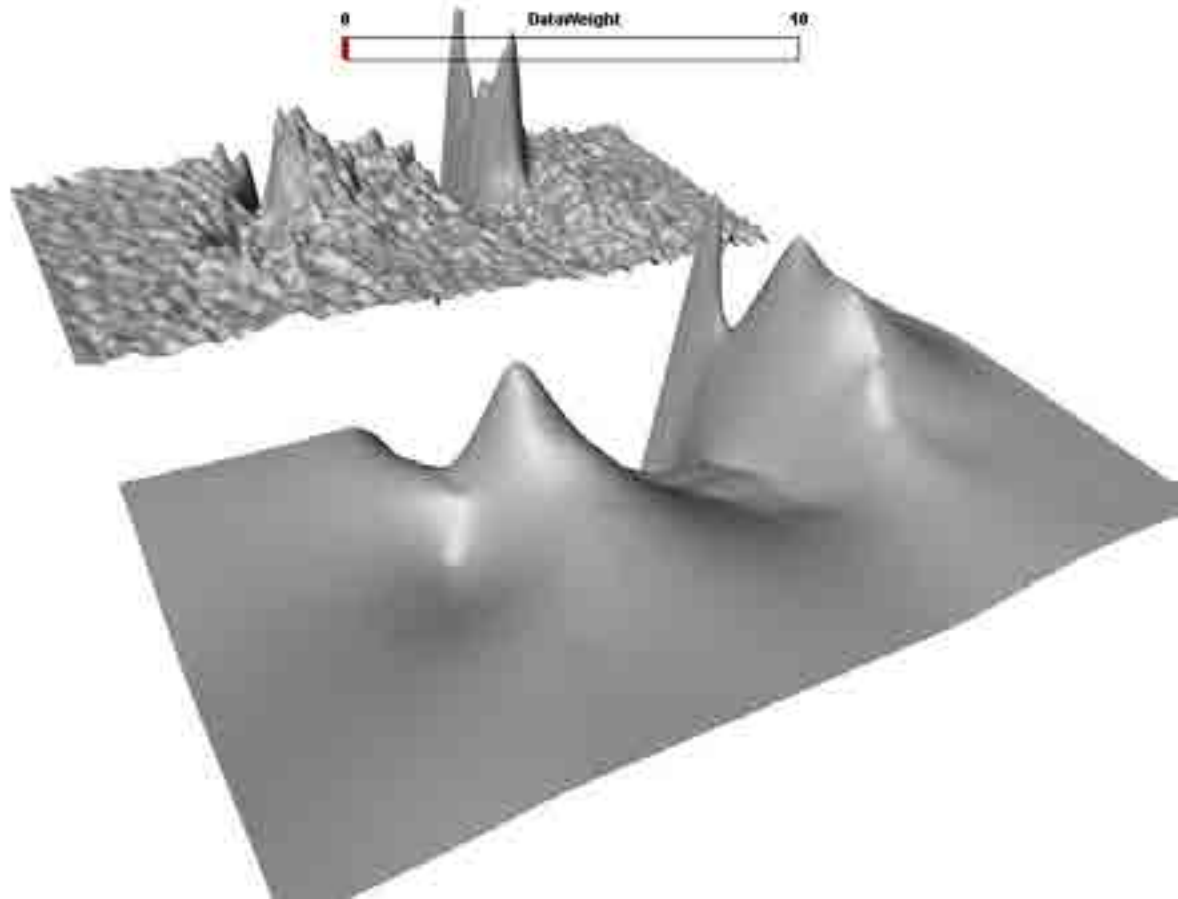


Simplified and smoothed



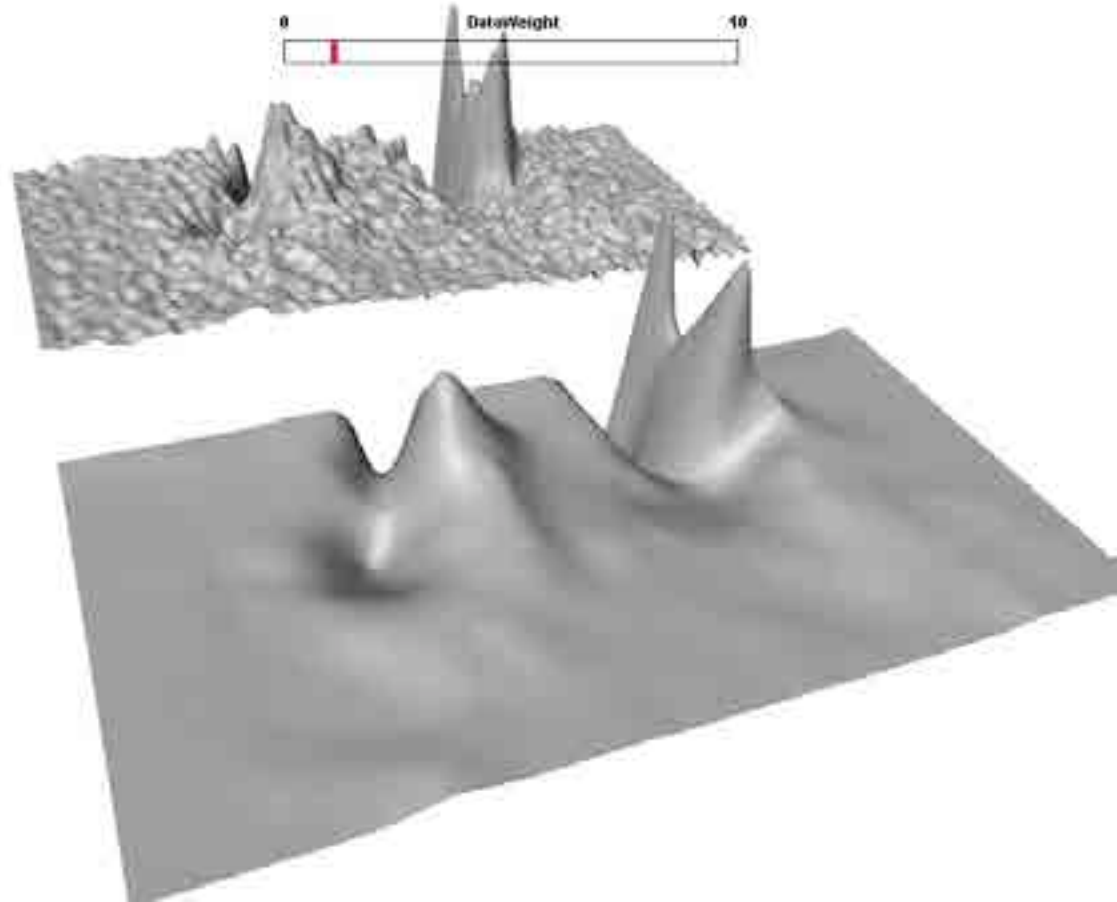
... but 1000 times faster

30K vertices
5 seconds per solve



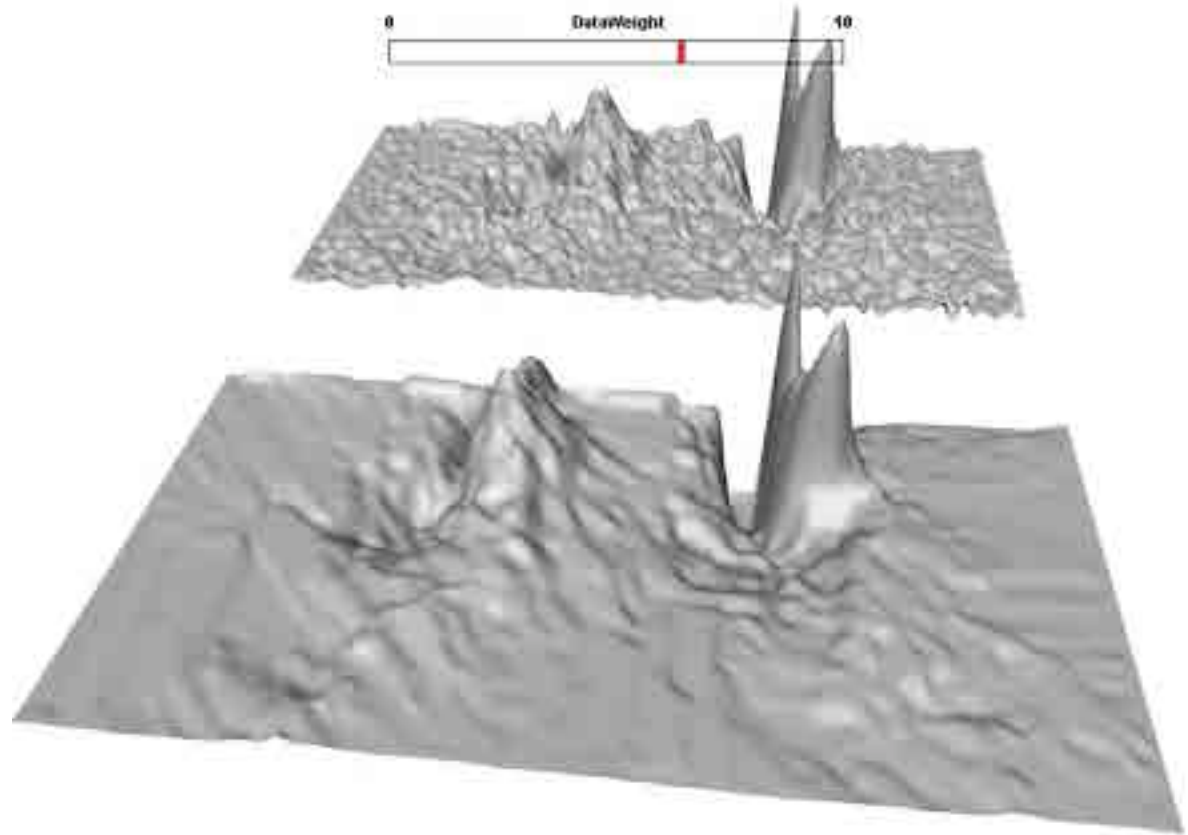
... but 1000 times faster

30K vertices
5 seconds per solve



... but 1000 times faster

30K vertices
5 seconds per solve



Conclusion: Important to control extrema

- Copy “monotonicity” of harmonic functions
- *Reduces* search-space, but optimization is tractable

Future work and discussion

- Larger, but still tractable subspace?
 - Consider all valid harmonic functions?

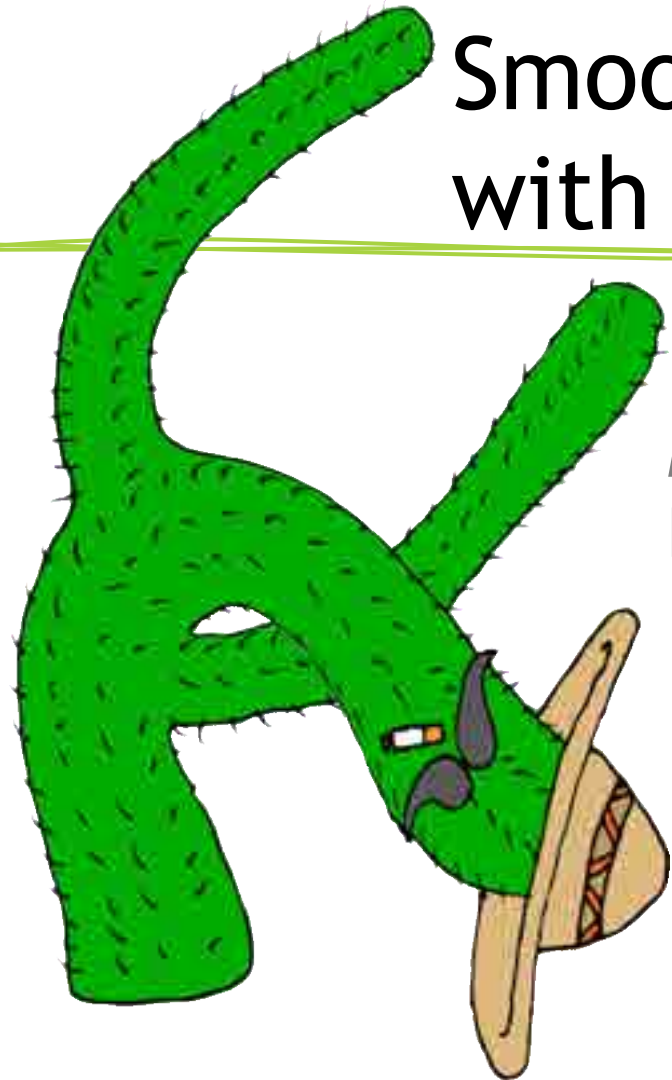
Future work and discussion

- Larger, but still tractable subspace?
 - Consider all valid harmonic functions?
- Continuous formulation?

Acknowledgements

We thank Kenshi Takayama for his valuable feedback. This work was supported in part by an SNF award 200021_137879 and by a gift from Adobe Systems.

Smooth Shape-Aware Functions with Controlled Extrema



MATLAB Demo:

<http://igl.ethz.ch/projects/monotonic/>

Alec Jacobson (jacobson@inf.ethz.ch)

Tino Weinkauf

Olga Sorkine